BOARD OF STUDIES
NEW SOUTH WALES

MATHEMATICS
2/3 UNIT
Years 11–12

◆

Syllabus

First published 1982
Reprinted November 1997
Revised (for web) August 1998
Revised (for web) May 2008

Published by
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Contents

Course Description (2 Unit)

The Mathematics 2 Unit Syllabus has been divided into a Preliminary course and an HSC course as follows:

Preliminary Course	HSC Course
Basic arithmetic and algebra (1.1–1.4)	Coordinate methods in geometry (6.8)
Real functions (4.1–4.4)	Applications of geometrical properties (2.5)
Trigonometric ratios (5.1–5.5)	**Geometrical applications of differentiation** (10.1–10.8)
Linear functions (6.1–6.5, 6.7)	**Integration** (11.1–11.4)
The quadratic polynomial and the parabola (9.1–9.5)	**Trigonometric functions** (including applications of trigonometric ratios) (13.1–13.6, 13.7)
Plane geometry – geometrical properties (2.1–2.4)	**Logarithmic and exponential functions** (12.1–12.5)
Tangent to a curve and derivative of a function (8.1–8.9)	**Applications of calculus to the physical world** (14.1–14.3)
	Probability (3.1–3.3)
	Series (7.1–7.3) and **Series applications** (7.5)

Note: Numbers given are syllabus references.

Course Description (3 Unit)

The Mathematics 3 Unit Syllabus has been divided into a Preliminary course and an HSC course as follows:

Preliminary Course	HSC Course
Other inequalities (1.4 E)	Methods of integration (11.5)
Circle geometry (2.6–2.10)	Primitive of $\sin 2x$ and $\cos 2x$ (13.6 E)
Further trigonometry (sums and differences, t formulae, identities and equations) (5.6–5.9)	Equation $\frac{dN}{dt} = k(N - P)$ (14.2 E)
Angles between two lines (6.6)	Velocity and acceleration as a function of x (14.3 E)
Internal and external division of lines into given ratios (6.7 E)	Projectile motion (14.3 E)
Parametric representation (9.6)	Simple harmonic motion (14.4)
Permutations and combinations (18.1)	Inverse functions and inverse trigonometric functions (15.1–15.5)
Polynomials (16.1–16.3)	Induction (7.4)
Harder applications of the Preliminary 2 Unit course	Binomial theorem (17.1–17.3)
	Further probability (18.2)
	Iterative methods for numerical estimation of the roots of a polynomial equation (16.4)
	Harder applications of HSC 2 Unit topics (including 10.5 E, 13.4 E, 14.1 E)

Notes

Applications of geometrical properties (from the HSC 2 Unit course) will need to be taught before the Preliminary 3 Unit Circle Geometry topic.

Numbers given are syllabus references.

Mathematics Syllabus
3 Unit and 2 Unit Courses

Introduction

The Board recognises that the aims and objectives of the syllabus may be achieved in a variety of ways and by the application of many different techniques. Success in the achievement of these aims and objectives is the concern of the Board which does not, however, either stipulate or evaluate specific teaching methods.

The 2 Unit Course

The content and depth of treatment of this course as specified in Part A and Part B indicate that it is intended for students who have completed the School Certificate mathematics course and demonstrated general competence in all the skills included in that course.

The 2 Unit course is intended to give these students an understanding of and competence in some further aspects of mathematics which are applicable to the real world.

The course has general educational merit and is also useful for concurrent studies in science and commerce. It is a sufficient basis for further studies in mathematics as a *minor* discipline at tertiary level in support of courses such as the life sciences or commerce. Students who require substantial mathematics at a tertiary level supporting the physical sciences, computer science or engineering should undertake the 3 or 4 Unit courses.

The 3 Unit Course

The content of this course, *which includes the whole of the 2 Unit course*, and its depth of treatment as specified in Part A and Part B indicate that it is intended for students who have demonstrated a mastery of the skills included in the School Certificate mathematics course and who are interested in the study of further skills and ideas in mathematics.

The 3 Unit course is intended to give these students a *thorough* understanding of, and competence in, aspects of mathematics including many which are applicable to the real world.

The course has general educational merit and is also useful for concurrent studies of science, industrial arts and commerce. It is a *recommended minimum* basis for further studies in mathematics as a *major* discipline at a tertiary level, *and for the study of mathematics in support of the physical and engineering sciences. Although the 3 Unit course is sufficient for these purposes, it is recommended that students of outstanding mathematical ability should consider undertaking the 4 Unit course.*

Objectives

Specific objectives of the course are:

(a) to give an understanding of important mathematical ideas such as variable, function, limit, etc, and to introduce students to mathematical techniques which are relevant to the real world;

(b) to understand the need to prove results, *to appreciate* the role of deductive reasoning in establishing such proofs, and to develop the ability to construct these proofs;

(c) to enhance those mathematical skills required for further studies in *mathematics, the physical sciences and the technological sciences*.

For achievement of these aims, the following points are important:

(i) Understanding of the basic ideas and precise use of language must be emphasised;

(ii) A clear distinction must be made between results which are proved, and results which are merely stated or made plausible;

(iii) Where proofs are given, they should be carefully developed, with emphasis on the deductive processes used;

(iv) Attaining competence in mathematical skills and techniques requires many examples, given as teaching illustrations and as exercises to be undertaken independently by the student;

(v) Since the course is to be 'useful for concurrent studies of science, industrial arts and commerce' students could be given some experience in applying mathematics to problems drawn from such areas. Realistic problems should follow the attainment of skills, and techniques of problem solving should be continually developed.

Scope and organisation of the syllabus

This syllabus is constructed on the assumption that students have acquired competence in the various mathematical skills related to the content of the mathematics course for the School Certificate. In particular it is expected that some familiarity with the material specified in the first few topics will have been gained. Nevertheless, the content of all topics listed in this syllabus is expected to be covered in the teaching of the course.

The order of topics in this syllabus is an indication of the connections among them, but is not prescriptive. Teachers are advised to familiarise themselves with the syllabus as a whole before planning a teaching program.

Part B of this syllabus is written for teachers and is intended to clarify the mathematical ideas underlying the whole syllabus and the various topics and to indicate the depth of treatment required. The methods and examples contained in them are not intended as a paradigm; it is the responsibility of each teacher to decide on matters such as the method of presentation of a topic and the setting out of examples.

The syllabus for the 2 Unit course is wholly contained in the following syllabus and consists of all those items not preceded by the letter 'E'. All 'E' (for 'Extension') items have been enclosed within 'boxes' for ease of identification. The 3 Unit course syllabus is the entire syllabus, and the 3 Unit student may be required to tackle harder problems on 2 Unit topics. A deeper treatment of common material is often appropriate for 3 Unit students.

All proofs given in the syllabus are expected to be discussed and treated as a normal part of the exposition, except where Part B indicates a lighter treatment. Students are not required to reproduce proofs of results contained in items preceded by the symbol † *except where Part B indicates that 3 Unit students are expected to be able to do so*.

It is assumed that electronic calculators will be available and used throughout the course.

Part A – Statement of Syllabus Topics

Explanation of symbols

†: denotes that students are *not* required to reproduce proofs of results contained in items preceded by this symbol, except where Part B indicates that 3 Unit candidates are expected to be able to do so.

E: denotes that the following item or items (enclosed within a box) are *not* included in the 2 Unit course.

1. Basic Arithmetic and Algebra

1.1 Review of arithmetical operations on rational numbers and quadratic surds.

† 1.2 Inequalities and absolute values.

1.3 Review of manipulation of and substitution in algebraic expressions, factorisation, and operations on simple algebraic fractions.

1.4 Linear equations and inequalities. Quadratic equations. Simultaneous equations.

E Other inequalities.

2. Plane Geometry

2.1 Preliminaries on diagrams, notation, symbols and conventions.

2.2 Definitions of special plane figures.

† 2.3 Properties of angles at a point and of angles formed by transversals to parallel lines. Tests for parallel lines.

Angle sums of triangles, quadrilaterals and general polygons.

Exterior angle properties.

Congruence of triangles. Tests for congruence.

Properties of special triangles and quadrilaterals. Tests for special quadrilaterals.

Properties of transversals to parallel lines.

Similarity of triangles. Tests for similarity.

Pythagoras' theorem and its converse.

Area formulae.

2.4 Application of above properties to the solution of numerical exercises requiring one or more steps of reasoning.

2.5 Application of above properties to simple theoretical problems requiring one or more steps of reasoning.

E
2.6 Harder problems extending 2.4 and 2.5.

2.7 Definitions of terms related to circles.

† 2.8 Simple angle properties of a circle.

2.9 Derivation of further angle, chord and tangent results.

2.10 Applications of 2.2, 2.3, 2.7, 2.8 and 2.9 to numerical and theoretical problems requiring one or more steps of reasoning.

3. Probability

3.1 Random experiments, equally likely outcomes;
probability of a given result.

3.2 Sum and product of results.

3.3 Experiments involving successive outcomes;
tree diagrams.

4. Real Functions of a Real Variable and their Geometrical Representation

4.1 Dependent and independent variables. Functional notation. Range and domain.

4.2 The graph of a function. Simple examples.

4.3 Algebraic representation of geometrical relationships.
Locus problems.

4.4 Region and inequality. Simple examples.

5. Trigonometric Ratios – Review and Some Preliminary Results

† 5.1 Review of the trigonometric ratios, using the unit circle.

† 5.2 Trigonometric ratios of: $-\theta$, $90° - \theta$, $180° \pm \theta$, $360° \pm \theta$.

5.3 The exact ratios.

5.4 Bearings and angles of elevation.

† 5.5 Sine and cosine rules for a triangle. Area of a triangle, given two sides and the included angle.

E
5.6 Harder applications of 5.3, 5.4 and 5.5.

5.7 Trigonometric functions of sums and differences of angles.

5.8 Expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of $\tan\left(\frac{\theta}{2}\right)$.

5.9 Simple trigonometric identities and equations.
The general solution of trigonometric equations.

6. Linear Functions and Lines

6.1 The linear function $y = mx + b$ and its graph.

6.2 The straight line: equation of a line passing through a given point with given slope; equation of a line passing through two given points; the general equation $ax + by + c = 0$; parallel lines; perpendicular lines.

6.3 Intersection of lines: intersection of two lines and the solution of two linear equations in two unknowns; the equation of a line passing through the point of intersection of two given lines.

6.4 Regions determined by lines: linear inequalities.

† 6.5 Distance between two points and the (perpendicular) distance of a point from a line.

E 6.6 The angle between two lines.

6.7 The mid-point of an interval.

E Internal and external division of an interval in a given ratio.

6.8 Coordinate methods in geometry.

7. Series and Applications

7.1 Arithmetic series. Formulae for the *n*th term and sum of *n* terms.

7.2 Geometric series. Formulae for the *n*th term and sum of *n* terms.

7.3 Geometric series with a ratio between –1 and 1. The limit of x^n, as $n \to \infty$, for $|x| < 1$, and the concept of limiting sum for a geometric series.

E 7.4 Mathematical induction. Applications.

7.5 Applications of arithmetic series.

Applications of geometric series: compound interest, simplified hire purchase and repayment problems.

Applications to recurring decimals.

8. The Tangent to a Curve and the Derivative of a Function

8.1 Informal discussion of continuity.

8.2 The notion of the limit of a function and the definition of continuity in terms of this notion. Continuity of $f + g$, $f - g$, fg in terms of continuity of f and g.

8.3 Gradient of a secant to the curve $y = f(x)$.

8.4 Tangent as the limiting position of a secant. The gradient of the tangent. Equations of tangent and normal at a given point of the curve $y = f(x)$.

8.5 Formal definition of the gradient of $y = f(x)$ at the point where $x = c$. Notations $f'(c)$, $\frac{dy}{dx}$ at $x = c$.

8.6 The gradient or derivative as a function. Notations $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$, y'

8.7 Differentiation of x^n for positive integral n. The tangent to $y = x^n$.

† 8.8 Differentiation of $x^{\frac{1}{2}}$ and x^{-1} from first principles. For the two functions u and v, differentiation of Cu (C constant), $u + v$, $u - v$, uv. The composite function rule. Differentiation of u/v.

† 8.9 Differentiation of: general polynomial, x^n for n rational, and functions of the form $\{f(x)\}^n$ or $f(x)/g(x)$, where $f(x)$, $g(x)$ are polynomials.

9. The Quadratic Polynomial and the Parabola

9.1 The quadratic polynomial $ax^2 + bx + c$. Graph of a quadratic function. Roots of a quadratic equation. Quadratic inequalities.

9.2 General theory of quadratic equations, relation between roots and coefficients. The discriminant.

9.3 Classification of quadratic expressions; identity of two quadratic expressions.

9.4 Equations reducible to quadratics.

9.5 The parabola defined as a locus. The equation $x^2 = 4Ay$. Use of change of origin when vertex is not at (0, 0).

E | 9.6 Parametric representation. Applications to problems concerned with tangents, normals and other geometric properties.

10. Geometrical Applications of Differentiation

10.1 Significance of the sign of the derivative.

10.2 Stationary points on curves.

10.3 The second derivative. The notations $f''(x)$, $\frac{d^2y}{dx^2}$, y''.

10.4 Geometrical significance of the second derivative.

10.5 The sketching of simple curves.

10.6 Problems on maxima and minima.

10.7 Tangents and normals to curves.

10.8 The primitive function and its geometrical interpretation.

11. Integration

† 11.1 The definite integral.

† 11.2 The relation between the integral and the primitive function.

† 11.3 Approximate methods: trapezoidal rule and Simpson's rule.

11.4 Applications of integration: areas and volumes of solids of revolution.

E 11.5 Methods of integration, including reduction to standard forms by very simple substitutions.

12. Logarithmic and Exponential Functions

12.1 Review of index laws, and definition of a^r for $a > 0$, where r is rational.

† 12.2 Definition of logarithm to the base a. Algebraic properties of logarithms and exponents.

† 12.3 The functions $y = a^x$ and $y = \log_a x$ for $a > 0$ and real x. Change of base.

† 12.4 The derivatives of $y = a^x$ and $y = \log_a x$. Natural logarithms and exponential function.

12.5 Differentiation and integration of simple composite functions involving exponentials and logarithms.

13. The Trigonometric Functions

13.1 Circular measure of angles. Angle, arc, sector.

13.2 The functions $\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$, $\cot x$ and their graphs.

13.3 Periodicity and other simple properties of the functions $\sin x$, $\cos x$ and $\tan x$.

13.4 Approximations to $\sin x$, $\cos x$, $\tan x$, when x is small.

The result $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

† 13.5 Differentiation of $\cos x$, $\sin x$, $\tan x$.

13.6 Primitive functions of $\sin x$, $\cos x$, $\sec^2 x$.

E | Primitive functions of $\sin^2 x$ and $\cos^2 x$.

13.7 Extension of 13.2 – 13.6 to functions of the form $a \sin(bx + c)$, etc.

14. Applications of Calculus to the Physical World

14.1 Rates of change as derivatives with respect to time.

The notation $\dot{x}$, $\ddot{x}$, etc.

† 14.2 Exponential growth and decay; rate of change of population;

the equation $\frac{dN}{dt} = kN$, where k is the population growth constant.

E | The equation $\frac{dN}{dt} = k(N - P)$, where k is the population growth constant, and P is a population constant.

14.3 Velocity and acceleration as time derivatives. Applications involving:

(i) the determination of the velocity and acceleration of a particle given its distance from a point as a function of time;

(ii) the determination of the distance of a particle from a given point, given its acceleration or velocity as a function of time together with appropriate initial conditions.

E | Velocity and acceleration as functions of x.

Applications in one and two dimensions (projectiles).

E | 14.4 Description of simple harmonic motion from the equation

$$x = a \cos(nt + \alpha),\ a > 0,\ n > 0.$$

The differential equation of the motion.

E

15. Inverse Functions and the Inverse Trigonometric Functions

15.1 Discussion of inverse function. The functions $y = \log_a x$ and $y = a^x$ as inverse functions. The relation

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

15.2 The inverse trigonometric functions.

15.3 The graphs of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$.

15.4 Simple properties of the inverse trigonometric functions.

15.5 The derivatives of $\sin^{-1}(x/a)$, $\cos^{-1}(x/a)$, $\tan^{-1}(x/a)$, and the corresponding integrations.

E

16. Polynomials

16.1 Definitions of polynomial, degree, polynomial equation. Graph of simple polynomials.

16.2 The remainder and factor theorems.

16.3 The roots and coefficients of a polynomial equation.

16.4 Iterative methods for numerical estimation of the roots of a polynomial equation.

E

17. Binomial Theorem

17.1 Expansion of $(1 + x)^n$ for $n = 2, 3, 4 \ldots$
Pascal Triangle.
Proof of the Pascal Triangle relations.
Extension to the expansion $(a + x)^n$.

† 17.2 Proof by Mathematical Induction of the formula for

$^{n}C_{k}$ (also denoted by $\binom{n}{k}$).

17.3 Finite series and further properties of binomial coefficients.

E

18. Permutations, Combinations and Further Probability

18.1 Systematic enumeration in a finite sample space.

Definitions of $^{n}P_{r}$, $^{n}C_{r}$ (also written $\binom{n}{r}$).

18.2 Binomial probabilities and the binomial distribution.

Part B

Additional Information
on
Syllabus Content

The material in this part of the syllabus contains suggestions for teachers as to the appropriate depth of treatment of, and some approaches to, topics in the syllabus. These suggestions are not intended to be prescriptive or definitive, nor is the amount of material on any topic an indication of the proportion of available time to be devoted to it.

The examples used in this part are illustrations only and are not intended to be exclusive indicators of likely examination questions.

1. Basic Arithmetic and Algebra

It is expected that this topic will receive continual attention over the two years, particularly through applications in other topics. It is left to the teacher's discretion whether to begin treatment of this entire topic immediately or to treat sections as the need arises in conjunction with other topics.

A distinction should be made between exact answers such as $\frac{3}{7}$ or $\sqrt{2}$, and approximate answers which may be obtained when using tables or a calculator.

1.1 The following are included in this section.

(i) Addition, subtraction, multiplication and division of fractions and decimals.

(ii) Conversion of fractions (rational numbers) to decimals and percentages, and vice versa. Students should be aware that a fraction (rational number) can be expressed as a terminating or repeating decimal and that conversely such decimals represent rational numbers. Repeating decimals may be converted to fractions by the method of removing the first period.

Example: express $0.\dot{1}\dot{2}$ as a fraction.

$$\begin{aligned} \text{Let } x &= 0.\dot{1}\dot{2}, \\ \text{then } 100x &= 12.\dot{1}\dot{2} \\ &= 12 + x, \\ \text{so } x &= \tfrac{12}{99} \\ &= \tfrac{4}{33}. \end{aligned}$$

Generally, if $x = \dot{r}$ is a recurring decimal with period length P, then

$$10^P x = r + \dot{r}$$

$$\text{so } (10^P - 1)x = r.$$

The method should also be applied to numbers such as $3.5\dot{1}\dot{2}$.

An alternative method is given in Topic 7.

(iii) Determination of powers and roots, eg:

$$(2\tfrac{1}{3})^3;\ (0.8)^2;\ \sqrt{6\tfrac{1}{4}};\ \sqrt{1.44}.$$

(iv) Scientific notation and approximation. Interpretation of calculator output, rounding off to a given number of significant figures or decimal places, eg:

5 230 100 = 5 230 000 or 5.230×10^6 (correct to 4 significant figures),

0.052 073 = 0.0521 (correct to 4 decimal places)
= 0.052 07 (correct to 4 significant figures).

In order to give an answer to a series of computations to a given accuracy, at least one additional figure must be used at each step of the computation and in obtaining the final answer, before rounding off.

(v) Evaluation of expressions involving combinations of parentheses, powers, roots and the four operations, eg:

evaluate $\sqrt{(5^2 + 7^2)}$, correct to two decimal places;

find the exact value of $\dfrac{\frac{2}{5}+\frac{2}{3}}{1-\frac{4}{15}}$.

(vi) Quadratic surds — the four operations, with division done by rationalising the denominator, eg:

show that $\dfrac{4}{2+\sqrt{5}} - \dfrac{1}{9-4\sqrt{5}}$ is rational.

Surdic equations and square roots of binomial surds are not included in this syllabus.

1.2 Inequalities should be reviewed, especially the effect of multiplication and division by negative numbers.

The absolute value $| a |$ equals a for $a \geq 0$, and $-a$ for $a < 0$.

The result $| a | = | a | . | b |$ is important.

The result (the 'triangle inequality') $| a + b | \leq | a | + | b |$ should be derived.

The geometric interpretation of $| x |$ as the distance of x from the origin, and more generally, of $| x - y |$ as the distance between x and y (on the number line).

Simple graphs involving absolute values (see Topic 4.2).

1.3 Attention should be given to the following matters.

(i) **Simplification** by removal of grouping symbols and collecting like terms, eg:

$-5x - 3\,(2x + 1)$; $4\,(x^2 + 5x - 7) - 3\,(2x^2 - 7x + 1)$.

The addition, subtraction and multiplication of algebraic expressions, eg:

remove the parentheses from $(2x + 3)(x^2 + 5x + 2)$;
subtract $5x + 2y - 3$ from $x - 7y + 9$.

(ii) **Substitution**

Evaluation of expressions involving the four operations, powers and roots. Numbers substituted may be integers, fractions, decimals or surds, eg:

find the exact value of $t^4 - t^2 + 1$ when $t = 2\sqrt{3}$;

find the exact value of
$\frac{A^4C}{B^4}$ where $A = \left(\frac{2}{3}\right)^2$, $B = \left(\frac{4}{3}\right)^4$, $C = \left(\frac{8}{3}\right)^7$.

Problems involving substitution of numerical values into common formulae should be practised, eg:

given that $V = \pi r^2 h$, find the exact value of h, when $V = 10$, $r = 2$;

find t given that $s = ut + \frac{1}{2}at^2$, $a = 4$,

$u = -4$, and $s = 6$.

(iii) **Factorisation**

Common factor, eg: $5x^2 - 10x = 5x(x - 2)$.

Difference of two squares, eg:

$16x^2 - 1 = (4x + 1)(4x - 1)$.

Trinomials, eg: $t^2 - 4t + 4 = (t - 2)^2$;

$3x^2 + 4x - 7 = (3x + 7)(x - 1)$.

Grouping of terms to involve the other types of factorisation, eg:

$$ax + ay - cx - cy = a(x + y) - c(x + y) = (x + y)(a - c)$$

$$x^2 - y^2 + 2x - 2y = (x - y)(x + y) + 2(x - y) = (x - y)(x + y + 2).$$

The sum and difference of two cubes, eg:

$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.

(iv) **Algebraic Fractions**

Reduction, eg:

$$\frac{5a-b}{2b-10a};\ \frac{x^2-5x+6}{x-2}.$$

Multiplication and division, eg:

$$\frac{3}{a-2}\div\frac{a+3}{a^2-4}.$$

Addition and subtraction, eg:

$$\frac{2m-n}{3}-\frac{m-3n}{6};\ \frac{2}{x}-\frac{3}{x(x+2)}.$$

1.4 The following are included in this section.

(i) **Linear equations**, such as:

$$5t+3=2(1-t);\ \frac{3x+4}{x}=2;$$

$$\frac{3x-1}{5x+1}=\frac{3x-2}{5x+2}.$$

(ii) **Linear inequalities**, their solution and description on a number line, including problems involving absolute values, but not with the unknown in a denominator, eg:

find the values of x for which:

(a) $3x+4>2\frac{1}{2}$; (b) $3-2x\le -1$; (c) $|x-1|<2$.

(iii) **Quadratic equations**, including solution by factorisation and by formula, eg:

$5x^2-11x+2=0$; $8t^2=1-10t$.

$y^2=6y$; $(v-2)^2=16$.

(iv) **Simultaneous equations**, only to the extent required by later topics. Students should always check the results by direct substitution in the original equations.

E (v) **Other inequalities**. 3 Unit students will be expected to be able to solve inequalities such as:

$$\frac{x^2-1}{x}>0;\ \frac{2t+1}{t-2}>1.$$

2. Plane Geometry

A knowledge of the various common geometrical figures and their properties is expected of all students. Further objectives in this topic are the development of understanding of the notions of congruence and similarity, the use of tests for congruence or similarity and the ability to use simple information combined with a few steps of reasoning to deduce additional information.

Students will be expected to be able to understand and use sketch diagrams and information shown on them. They will also be expected to be able to draw sketch diagrams from given data.

It is anticipated that the contents of this topic will be reviewed if changes occur in the teaching of geometry in the junior school.

2.1 Standard notation for points, lines and angles (including $\angle ABC$, $A\hat{B}C$, $\hat{B}$) should be known, as should the meanings of phrases like right angle, collinear points, foot of perpendicular etc. In describing a quadrilateral or polygon, the vertices should be given in cyclic order. The notation AB may be used to describe a line, line segment (interval), ray or the length of the corresponding interval, and the context used to determine meaning. For example, $AB = CD$ is a reference to lengths and $AB \perp CE$ is a reference to lines or intervals. The statement 'X is a point on AB produced' means that the line segment AB is extended beyond B and that X is some point on this extension.

Acceptable symbols include the following:

is parallel to	$\parallel$	*is similar to*	$\vert\vert\vert$
is perpendicular to	$\perp$	*therefore*	$\therefore$
is congruent to	$\equiv$	*because*	$\because$

2.2 Definitions should be given of isosceles and equilateral triangles and of quadrilaterals including the standard special types (parallelogram, rectangle, square, rhombus and trapezium). A parallelogram should be defined as a quadrilateral with both pairs of opposite sides parallel and a rectangle as a parallelogram with one angle a right-angle. A rhombus is a parallelogram with a pair of adjacent sides equal and a square is a rectangle with a pair of adjacent sides equal. A regular polygon should be defined as a polygon with all sides equal and all angles equal.

2.3 It is expected that the material in this section will be developed in a logical order, with proofs being provided both for general results and for all examples. The setting out of arguments (information given, results to be proved and steps in the argument with reasons for them) is to be demonstrated frequently and practice given in its use. Any result may be used by students in the proofs of subsequent exercises as long as specific mention is made of it in the solution of the exercise.

Properties that students should encounter include the following:

- Properties of angles on a straight line, vertically opposite angles, angles at a point.

- Definitions of parallel lines, transversal, alternate, corresponding and cointerior (allied) angles. Properties of these angles. Tests for parallel lines. The fact that if two lines are parallel to a third line then they are parallel to one another.
- Exterior angle and angle sum of a triangle. Angle sum of a quadrilateral and of a general polygon.
- The size of the angles of a general polygon.
- Sum of the exterior angles of a general polygon.
- Definition of congruence of triangles. Statement of tests for congruence, including test for right-angled triangles.
- Properties of isosceles and equilateral triangles.
- Properties of quadrilaterals:
  - (a) parallelogram (equality of opposite sides and angles, bisection of diagonals). Tests for parallelograms (both pairs of opposite sides parallel, both pairs of opposite angles equal, one pair of opposite sides equal and parallel, diagonals bisect each other).
  - (b) rhombus (diagonals bisect each other at right angles, diagonals bisect the angles through which they pass). Tests for a rhombus (all sides equal, diagonals bisect at right angles).
  - (c) rectangle (diagonals are equal).
  - (d) square.
- The intercept properties of transversals to parallel lines (if a family of parallel lines cuts equal intercepts on one transversal, it does so on all transversals).
- Definition of similarity of triangles. Statement of tests for similarity of triangles (equality of corresponding angles or of two such pairs, corresponding sides proportional, two pairs of corresponding sides proportional and equality of the included angles). Parallel lines preserve ratios of intercepts on transversals. Line parallel to one side of a triangle divides the other two sides in proportion*.
- A line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.*
- Pythagoras' Theorem. Proof using similar triangles. Converse. Area formulae for parallelogram, triangle, trapezium and rhombus.

* Note: These properties are simple consequences of other listed properties. They may be derived and quoted in proofs of exercises but are not essential parts of the syllabus.

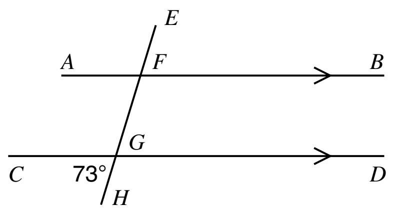
2.4, 2.5, 2.6 Problems for both 2 and 3 Unit students may involve the application of any of the properties treated above. For 2 Unit students, problems should mainly have diagrams supplied although practice should be given in sketching a diagram from a given set of data. Problems may be either numerical or general, with the first type being stressed for 2 Unit students. In all cases, a geometrical justification for each step will be required where it is appropriate.

Sample exercises with solutions are supplied in these notes as an indication of the clarity of the argument that needs to be demonstrated.

The exercises which students should practise may be of several types.

(i) **Simple numerical**

Example 1



Find $E\hat{F}B$ if $AB \parallel CD$.

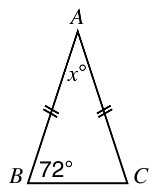
Solution

In the figure, $AB \parallel CD$.

$\therefore\ A\hat{F}G = 73^\circ$ (corresponding to $C\hat{G}H$).

$\therefore\ E\hat{F}B = 73^\circ$ (vertically opposite to $A\hat{F}G$).

Example 2



$AB = AC$. Find x.

Solution

Since $AB = AC$ then ΔABC is isosceles.

$\therefore\ A\hat{C}B = 72^\circ$.

$x + 144 = 180$ (angle sum of Δ).

$\therefore\quad x = 36$.

Example 3

Find the size of an interior angle of a regular hexagon.

Solution

Angle sum of an n-sided polygon = $(2n - 4)$ right angles.

$\therefore$ Angle sum of a hexagon = 8 right angles

= 720˚.

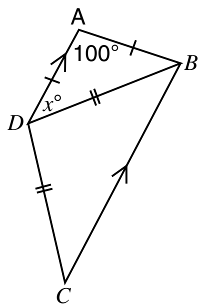
$\therefore$ Size of each angle = 720˚ ÷ 6

= 120˚.

(ii) **Numerical but involving a larger number of steps**

Example 1

Given $AD = AB$, $DB = DC$ and $AD \parallel BC$, find $\angle BDC$.



Solution

Let $\angle ADB = x^\circ$.

ΔADB is isosceles ($AB = AD$).

$\therefore$ $\angle ADB = \angle ADB$ (base angles of ΔADB).

Then $2x + 100 = 180$ (angle sum of ΔADB).

So $x = 40$,

$\therefore$ $\angle ADB = 40^\circ$.

Then $\angle DBC = 40^\circ$ (alternate to $\angle ADB$, $AD \parallel BC$);
but ΔDBC is isosceles ($DB = DC$).

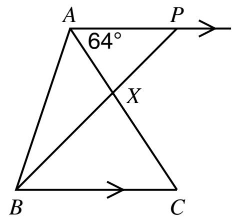
$\therefore$ $\angle DBC = 40^\circ$ (base angles of ΔDBC).

$\therefore$ $\angle BDC = 180^\circ - 40^\circ - 40^\circ$ (angle sum of ΔDBC)

$= 100^\circ$.

Example 2

In the figure, $AP \parallel BC$ and $AB = AC$, the lines BP and AC meet at right angles at X, with $\angle PAC = 64°$.



(i) Calculate $\angle ABC$.

(ii)Find $\angle APB$ and deduce that $\angle ABP = 38°$.

Solution

(i) $\angle ACB = 64°$ (alternate to $\angle PAC$, $AP \parallel BC$).

$\therefore\ \angle ABC = 64°$ (base angles of isosceles ΔABC).

(ii) $\angle APB + 90° + 64° = 180°$ (angle sum of ΔAPX).

$\therefore\ \angle APB = 26°$.

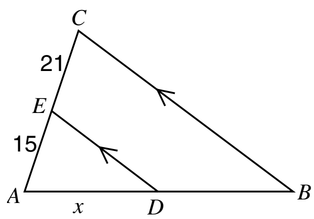
$\angle PBC = 26°$ (alternate to $\angle APB$, $AP \parallel BC$).

$\therefore\ \angle APB = 64° - 26°$

$= 38°$

Example 3

In the figure, $AE = 15$ cm, $AB = 24$ cm, $EC = 21$ cm and $DE \parallel BC$. Find the length of AD.



Solution 1

In Δs ADE, ABC,

$\angle ADE = \angle ABC$ (corresponding angles, $DE \parallel BC$),

$\angle DAE = \angle BAC$ (common angle).

$\therefore\ \Delta ADE \ ||| \ \Delta ABC$.

Let $AD = x$ cm.

$\therefore\ \dfrac{AD}{AB} = \dfrac{AE}{AC}$

$\therefore\ \dfrac{x}{24} = \dfrac{15}{36}$

$\therefore\ x = 10$.

Solution 2

$\frac{AE}{EC} = \frac{AD}{DB}$ (ratio property of parallel lines)

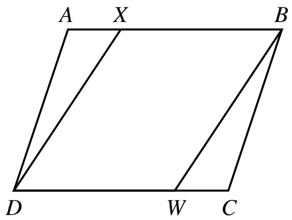
$\frac{15}{21} = \frac{x}{(24 - x)}$

$\therefore\ 360 - 15x = 21x$

$x = 10.$

(iii) **Simple deductions (with or without the diagram supplied)**

Example 1



ABCD is a parallelogram,

$AX = CW$.

Prove $XD = BW$.

Proof In ΔAXD, ΔBCW

$AX = CW$ (given),

$AD = BC$ (opposite sides of parallelogram are equal),

$X\hat{A}D = B\hat{C}W$ (opposite angles of parallelogram are equal)

$\therefore\ AXD \equiv \Delta CWB$ (*S.A.S.*).

$\therefore\ XD = BW$.

E (iv) **Harder deductive exercises (with or without diagram supplied)**

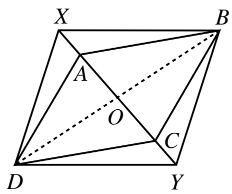
Example

ABCD is a parallelogram,

AC is produced to *Y* and *CA* to *X*

such that $AX = CY$.

Prove that *XBYD* is a parallelogram.



Proof Join diagonal *BD*. Let *BD* and *AC* intersect at *O*.

Since the diagonals of the parallelogram *ABCD* bisect each other at *O*, $DO = OB$ and $AO = OC$.

But $AX = CY$ (given),

$\therefore\ OX = OY$ (by addition).

In quadrilateral *XBYD*, diagonals *XY* and *BD* bisect each other.

$\therefore$ *XBYD* is a parallelogram.

E

2.7 Definitions of circle, centre, radius, diameter, arc, sector, segment, chord, tangent, concyclic points, cyclic quadrilateral, an angle subtended by an arc or chord at the centre and at the circumference, and of an arc subtended by an angle should be given.

Two circles touch if they have a common tangent at the point of contact.

2.8 Assumption: equal arcs on circles of equal radii subtend equal angles at the centre, and conversely.

The following results should be discussed and proofs given. Reproduction of memorised proofs will not be required.

Equal angles at the centre stand on equal chords. Converse.

The angle at the centre is twice the angle at the circumference subtended by the same arc.

The tangent to a circle is perpendicular to the radius drawn to the point of contact. Converse.

2.9 3 Unit students will be expected to be able to prove any of the following results using properties obtained in 2.3 or 2.8.

The perpendicular from the centre of a circle to a chord bisects the chord.

The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Equal chords in equal circles are equidistant from the centres.

Chords in a circle which are equidistant from the centre are equal.

Any three non-collinear points lie on a unique circle, whose centre is the point of concurrency of the perpendicular bisectors of the intervals joining the points.

Angles in the same segment are equal.

The angle in a semi-circle is a right angle.

Opposite angles of a cyclic quadrilateral are supplementary.

The exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.

If the opposite angles in a quadrilateral are supplementary then the quadrilateral is cyclic (also a test for four points to be concyclic).

If an interval subtends equal angles at two points on the same side of it then the end points of the interval and the two points are concyclic.

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

Tangents to a circle from an external point are equal.

The products of the intercepts of two intersecting chords are equal.

The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.

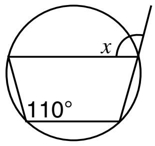
When circles touch, the line of centres passes through the point of contact.

2.10 In applications to problems, any of the definitions given or results obtained in 2.2, 2.3, 2.7, 2.8 or 2.9 may be used without proof, provided a specific reference is made to each result so used. If a proof is required for any of the results in 2.9 then this will be clearly indicated. The following exercises are given as examples of the types of problems to be discussed.

E (i) **Simple numerical**

Example 1

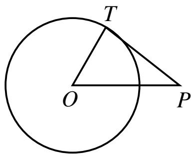
Find x.



Solution $x = 110^\circ$ (exterior angle of a cyclic quadrilateral equals interior opposite angle).

Example 2

TP is a tangent.
O is the centre.
OP is 10 cm and the radius is 6 cm.
Find PT.



Solution $O\hat{T}P = 90^\circ$ (angle between a tangent and radius).

Then $OT^2 + PT^2 = OP^2$ (Pythagoras' Theorem)

$36 + PT^2 = 100$

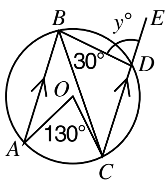
$\therefore\ PT = 8$

PT is 8 cm.

(ii) **Numerical but involving several steps**

Example

$AB \parallel CD$. O is the centre.
Find y.



Solution

$A\hat{B}C = 65^\circ$ (angle at centre is double angle at circumference).

$\therefore\ A\hat{B}D = 95^\circ$.

$B\hat{D}E = A\hat{B}D$ (alternate angles, $AB \parallel CD$).

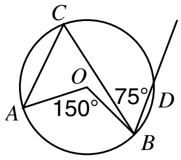
$y = 95$.

(iii) **Simple deductions with diagram supplied**

Example 1

O is the centre

Prove that $AC \parallel BD$.



E

Solution

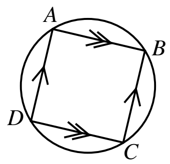
$A\hat{C}B = 75^\circ$ (angle at centre is double the angle at circumference).

$\therefore\ A\hat{C}B = C\hat{B}D$ (both 75°).

$\therefore$ $AC \parallel BD$ (since alternate angles are equal).

(iv) **Harder deductions (with or without supplied diagram)**

Example 1 A parallelogram *ABCD* is inscribed in a circle as shown. Prove that *ABCD* is also a rectangle.



Data: *ABCD* is a cyclic quadrilateral
$AB \parallel CD$, $AD \parallel BC$.

Aim: To prove *ABCD* is a rectangle.

Proof $\angle A = \angle C$ (opposite angles of a parallelogram);

but $\angle A$ is the supplement of $\angle C$ (*ABCD* is a cyclic quadrilateral).

$\therefore\ \angle A = 90^\circ$,

$\therefore$ the figure *ABCD* is a rectangle (parallelogram with an angle of 90°).

Note: There is no one correct form of setting out the reasons. Alternatives are given below. The important thing is that reasons are given and that steps follow a logical pattern.

Alternatively

Let the angle *A* be x°.

Because *ABCD* is a parallelogram,

$\angle C = \angle A = x^\circ$ (opposite angles equal).

Because *ABCD* is a cyclic quadrilateral,

$\angle C = 180^\circ - x^\circ$ (opposite angles supplementary).

$\therefore\ x^\circ = 180^\circ - x^\circ$,

$\therefore\ x^\circ = 90^\circ$.

Thus the figure *ABCD* is a rectangle.

Alternatively

Produce *DC* to *E*.

$\angle BCD = \angle BAD$ (opposite angle parallelogram),

$\angle BCE = \angle BAD$ (exterior angle cyclic quadrilateral),

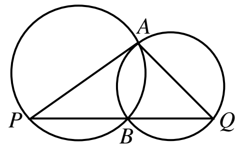
$\therefore\ \angle BCD = \angle BCE$.

But, because *DCE* is a straight line, each is a right angle. Hence the figure *ABCD* is a rectangle.

E

Example 2 Two circles intersect at A and B. AP and AQ are diameters in the respective circles. Prove that the points P, B, Q are collinear.

Data: AP and AQ are diameters.



Aim: To prove P, B and Q are collinear.

Proof Join AB.

$A\hat{B}P = 90^\circ$ (angle in a semicircle, AP diameter).

$A\hat{B}Q = 90^\circ$ (angle in a semicircle, AQ diameter).

PB and BQ form the one line (adjacent angles supplementary).

3. Probability

Students should be familiar with the common terms used in popular activities and games, hence examples should be given in pastimes such as playing cards, Monopoly and backgammon as well as in gaming activities such as lotteries and raffles, the tossing of coins and the throwing of dice.

3.1 Use everyday examples of ‘random experiments’, such as coin tossing, throwing dice, drawing raffles or lotteries, dealing cards, to introduce the ideas of *outcomes* of experiments, the notion of *equally likely* outcomes, and the idea that for experiments having a finite number n of *equally likely, mutually exclusive* outcomes $E_1, \ldots, E_n$, the probability $P(A)$ of a single *result* A is given by

$$P(A) = \frac{\text{number of outcomes that produce } A}{n}.$$

In particular, since one of $E_1, ..., E_n$ must occur,

$$P(E_1) + ... + P(E_n) = 1,$$

and for *any* result A,

$$0 \le P(A) \le 1.$$

Examples

(i) An ordinary die is thrown. Find the probabilities that (a) 1 is shown, (b) an odd number is shown.

Solution.

(a) Since the 6 outcomes are equally likely, the probability of the outcome '1 is shown' is 1/6.

(b) An odd number is shown if and only if any of the outcomes 1, 3, 5 occur. Hence the probability is 3/6 = 1/2.

(ii) A pair of dice are thrown. What is the probability that they show a total of three?

Solution. Each outcome of the first die is equally likely to occur with each outcome of the second die. The total number of possible outcomes is 6 × 6 = 36, each occurring with a probability 1/36. The outcomes producing a total of three and a 1 and a 2, or a 2 and a 1. Hence the probability of a total of three is $\frac{2}{36} = \frac{1}{18}$.

(iii) In a raffle, 30 tickets are sold and there is one prize. What is the probability that someone buying 5 tickets wins the prize?

Solution. The probability that a given ticket wins the prize is 1/30. Hence the probability of winning with 5 tickets is 5/30 = 1/6.

(iv) A card is drawn randomly from a standard pack of 52 cards. What is the probability that it is an even-numbered card?

Solution. Of the 52 equally likely outcomes, the drawing of a 2, 4, 6, 8 or 10 of clubs, diamonds, hearts or spades are the outcomes producing the result. The probability of the result is thus (5 × 4)/52 = 5/13.

Practice should be given in calculating the probabilities of various types of result (both composite and simple) from a knowledge of the probabilities of the possible outcomes of an experiment.

The complementary result $\overline{A}$ (A does *not* occur, or 'not A') to A should be defined, the relation $P(\overline{A}) + P(A) = 1$ derived, and use made of it in simple examples.

3.2 In the case of mutually exclusive outcomes A_1, A_2, the probability that A_1 or A_2 occurs is the sum of the probabilities that A_1, A_2 each occur. Denoting the result 'A_1 or A_2' by $A_1 \cup A_2$ (called the *sum* of A_1, and A_2),

$$P(A_1 \cup A_2) = P(A_1) + P(A_2), \qquad (1)$$

and generally, for mutually exclusive outcomes $A_1, \ldots, A_n$,

$$P(A_1 \cup \ldots \cup A_n) = P(A_1) + \ldots + P(A_n).$$

Sometimes, two results may occur together (ie they are not mutually exclusive). For example, in randomly selecting a digit from the digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, if

A is the result 'an even digit is selected' and B is the result 'a digit less than 5 is selected',

then A and B will both occur if 2 or 4 is selected. Denoting the result 'A and B', called the product of A and B, by AB,

$P(AB) = 2/9$.

$A \cup B$ occurs if 1, 2, 3, 4, 6 or 8 is selected, thus

$$P(A \cup B) = 6/9 = 2/3,$$

and in this experiment,

$$P(A \cup B) \neq P(A) + P(B) \ (= 4/9 + 4/9 = 8/9).$$

This inequality holds because A and B are not mutually exclusive results. For general results A, B, the formula (1) must be replaced by

$$P(A \cup B) = P(A) + P(B) - P(AB), \qquad (2)$$

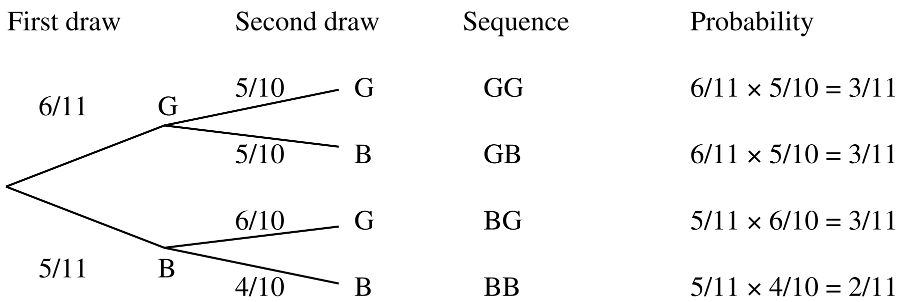
and it is readily verified that (2) holds in the above example. Practice in the use of (2) should be given, but formal proofs of (1) or (2) are not required.

3.3 Examples should be given which illustrate the difference between, say, successive tosses of a coin (where the probabilities of successive outcomes do not depend on previous outcomes) and the drawing of names from a hat (where probabilities depend on previous outcomes).

Tree diagrams should be used to trace the possible outcomes of two or three stage experiments, and hence to calculate the probabilities of certain final results. Explanation of all steps in the diagram should be given so that students can construct diagrams, as shown below, directly from given information.

Examples

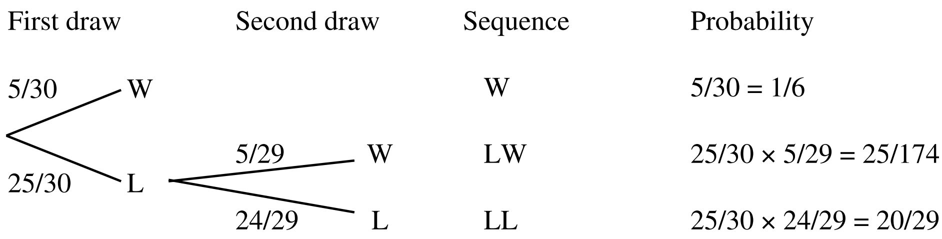
(i) 5 boys' names and 6 girls' names are in a hat. Find the probability that in two draws a boy's name and a girl's name are chosen. (No replacement of names after a draw.)



Required probability = $P(GB \cup BG) = 3/11 + 3/11 = 6/11$.

(ii) In a raffle, 30 tickets are sold and there are two prizes. What is the probability that someone buying 5 tickets wins at least one prize?

In this example, we simplify the tree diagram by considering carefully what it is we are required to find. The required result is obtained in exactly two exclusive ways: either first prize is won, in which case it does not matter whether second prize is won or not, or first prize is not won but second prize is won.



Required probability = $P(W \cup LW)$ = 1/6 + 25/174 = 9/29.

An alternative method is to notice that the probability of winning at least one prize is the complement of winning no prizes, ie:

Required probability = 1 – $P(LL)$

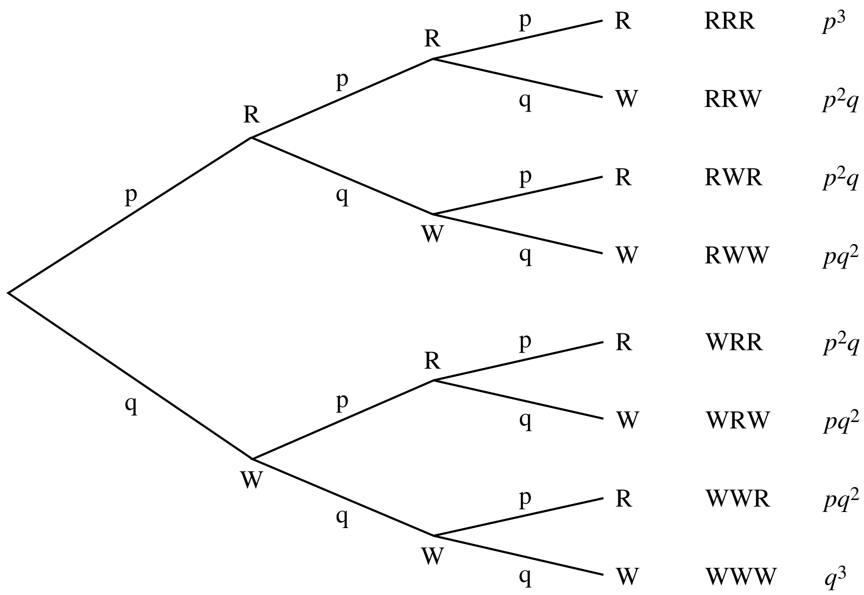
= 1 – 25/30 × 24/29

= 1 – 20/29

= 9/29.

(iii) In a mixture of red and white pebbles in a gravel, red and white pebbles occur in the ratio of 3 to 7. Find the probability that if 3 pebbles are chosen from the mixture, (a) exactly two are red; (b) at least one is white.

Let p (= 0.3) and q (= 0.7) be respectively the probabilities of choosing a red or a white pebble. The tree diagram is as follows:



(a) Required probability = $P(RRW \cup RWR \cup WRR) = 3p^2q$ = 189/1000

(b) Required probability = $P(RRW \cup RWR \cup RWW \cup WRR \cup WRW \cup WWR \cup WWW)$

$= 3p^2q + 3pq^2 + q^3$

= 973/1000.

In examples such as (b), it is important to realise that the result is more readily obtained by calculating the complementary probability (that none is white) and subtracting from 1. Here, the probability of obtaining 3 red pebbles is p^3, hence the probability of obtaining at least one white pebble is $1 - p^3 = 973/1000$.

4. Real Functions of a Real Variable and their Geometrical Representation

4.1 Much of this course is devoted to the study of properties of real-valued functions of a real variable. Such a function f assigns to *each* element x of a given set of real numbers *exactly one* real number y, called the *value* of the function f at x. The dependence of y on f and on x is made explicit by using the notation $f(x)$ to mean the value of f at x. The set of real numbers x on which f is defined is called the *domain* of f, while the set of values $f(x)$ obtained as x varies over the domain of f is called the *range* or *image* of f. x is called the *independent variable* since it may be chosen freely within the domain of f, while $y = f(x)$ is called the *dependent variable* since its value depends on the value chosen for x.

The functions f studied in this course are usually given by an explicit rule involving calculations to be made on the variable x in order to obtain $f(x)$. For this reason, a function f is often described in a form such as '$y = f(x)$' with the domain of x specified (eg '$y = x^2 + 1$, for $-1 \le x \le 1$' and referred to as 'the function $f(x)$' or as 'the function $x^2 + 1$, $-1 \le x \le 1$'.

It is also common usage to refer to 'the function $f(x)$' where $f(x)$ is prescribed but no domain is given. In such cases, the understanding required to be developed is that the domain of f is the set of real numbers for which the expression $f(x)$ defines a real number. For example, 'the function $\sqrt{(1 - x^2)}$' has domain the interval $-1 \le x \le 1$ and its value at x in this interval is $\sqrt{(1 - x^2)}$; 'the function $\frac{x}{x^2 - 1}$' has domain all real x except $x = \pm 1$; 'the function $\frac{x}{|x|}$' has domain all real x except $x = 0$; 'the function $x^2 + 1$' has domain all real x.

It is important to realise that use of the notation $y = f(x)$ does not imply that the expression corresponding to $f(x)$ is the same for all x. For example, the rule

$$f(x) = x \text{ for } x \ge 0,$$

$$f(x) = -x \text{ for } x < 0,$$

defines a function with domain all real x.

The use of x and y is customary and is related to the geometrical representation of a function f by graphing the set of points $(x, f(x))$ for x in the domain of f, using cartesian (x, y) coordinates. Other symbols for independent and dependent variables occur frequently in practice and students should become familiar with functions defined in terms of other symbols.

Examples should be chosen to illustrate the points made above and also with a view towards illustrating different types of behaviour of functions and their graphs, which are of importance in later work.

The concept of a function defined on an abstract set and formal definitions involving such functions are not required in this course.

4.2 The pictorial representation of a function is extremely useful and important, as is the idea that algebraic and geometrical descriptions of functions are both helpful in understanding and learning about their properties.

The function $y = f(x)$ may be represented pictorially by its graph, which is the set of points $(x, f(x))$ for each x in the domain of f, indicated with respect to cartesian coordinate axes $OxOy$. Denoting the point with coordinates $(x, f(x))$ by P, the graph of the function (and sometimes the function itself) is often referred to as the 'set of points $P(x, f(x))$'. Since f is a function, there is at most one point P of its graph on any ordinate. The graph of $y = f(x)$ is also called the *curve* $y = f(x)$ and the part of the curve lying between two ordinates is called an *arc*.

Examples of functions $y = f(x)$ should be given which illustrate different types of domain, bounded and unbounded ranges, continuous and discontinuous curves, curves which display simple symmetries, curves with sharp corners and curves with asymptotes.

Students are to be encouraged to develop the habit of drawing sketches which indicate the main features of the graphs of any functions presented to them. They should also develop at this stage the habit of checking simple properties of functions and identifying simple features such as: where is the function positive? negative? zero?; where is it increasing? decreasing?; does it have any symmetry properties?; is it bounded?; does it have gaps (jumps) or sharp corners?; is there an asymptote?

Knowledge of the symmetries of the graphs of odd and even functions is useful in curve sketching.

A function $f(x)$ is *even* if $f(-x) = f(x)$ for all values of x in the domain. Its graph is symmetric with respect to reflection in the y-axis, ie it has line symmetry about the y-axis.

A function $f(x)$ is *odd* if $f(-x) = -f(x)$ for all values of x in the domain. Its graph is symmetric with respect to reflection in the point O (the origin or axes), ie it has point symmetry about the origin.

4.3 Some of the work of this section might profitably be discussed in conjunction with Topics 6 and 9. A circle with a given centre C and a given radius r is defined as the set of points in the plane whose distance from C is r. If cartesian coordinate axes $OxOy$ are set up in the plane so that C is the point with coordinates (a, b), then the distance formula shows that $P(x, y)$ lies on the given circle if and only if x and y satisfy the equation $(x - a)^2 + (y - b)^2 = r^2$, hence this equation is an algebraic representation corresponding to the geometrical description given above.

It should be noted that if this equation is used to express y as a *function* of x, then two functions are obtained: $y = b + \sqrt{(r^2 - (x - a)^2)}$ and $y = b - \sqrt{(r^2 - (x - a)^2)}$, each with domain $a - r \le x \le a + r$.

Generally, sets of points satisfying simple conditions stated in geometrical terms can be described in algebraic terms by introducing cartesian coordinates and interpreting the original conditions as conditions relating x and y. The conditions then usually reduce to one or more equations or inequalities.

Problems involving the determination of the set of points which satisfy a given number of conditions (which may be expressed geometrically or algebraically) are called *locus* problems and often stated in the form 'Find the locus of a point P which satisfies …'. This means in practice 'Find a simple algebraic or geometric description of the set of all points P which satisfy …'.

Examples

(i) The locus of all points P in the plane which are at a distance $r > 0$ from the point $C(a, b)$ is the circle centre C, radius r. Its equation is

$$(x - a)^2 + (y - b)^2 = r^2.$$

(ii) The locus whose equation is $x^2 - y^2 = 0$ consists of the points lying on either of the straight lines $y = x$, $y = -x$.

(iii) The locus of points equidistant from two distinct points A and B is the perpendicular bisector of the segment AB. To find its equation, choose axes so that A is $(a, 0)$ and B is $(-a, 0)$. The condition that $P(x, y)$ satisfies $AP = PB$, and so $AP^2 = PB^2$, is

$$(x - a)^2 + y^2 = (x + a)^2 + y^2$$

which reduces to $x = 0$, which is the equation of the y-axis.

(iv) A *parabola* may be defined as the locus of a point whose distance from a given fixed point (its *focus*) equals its distance from a given fixed line (its *directrix*). Choosing coordinates so that the focus is $(0, A)$ and the directrix has equation $y = -A$, the equation of the parabola reduces to $x^2 = 4Ay$. The terms *focus, directrix, vertex, axis* and *focal length* should be defined and illustrated with many examples of parabolas. For any parabola with a vertical or a horizontal axis, students should be able to derive its coordinate equation.

In locus problems and in problems involving curves or functions, students should be required to produce sketch diagrams showing clearly the main features of the locus, curve or function.

The centre and radius of the circle

$x^2 + y^2 + 2ax + 2by + c = 0$ should be found, as should the vertex and focus of the parabola $y = ax^2 + bx + c$.

4.4 Treatment is to be restricted to regions of the (cartesian x, y–) plane which admit a simple geometrical description – for example, by use of words such as *interior, exterior, bounded by, boundary, sector, common to*, etc – and which admit a simple algebraic description using one or more inequalities in x and y.

Examples should be simple and involve at most one non-linear inequality, but should include both bounded and unbounded regions. Note that the case of one or more linear inequalities is specifically listed in Topic 6.4.

A clear sketch diagram, illustrating the relevant regions, should be drawn for each example.

Regions whose algebraic description involves two or more inequalities should be understood to correspond to the common part (intersection) of the regions determined by each separate inequality.

Examples

(i) Indicate using a clear sketch diagram the region determined by the inequality

$(x-3)^2 + y^2 > 1$.

(ii) Find a system of inequalities in x and y whose solutions correspond exactly to the points of the region of the cartesian x, y-plane lying inside or on the circle with centre (0,0) and radius 3 but to the right of the line $x = 2$. Sketch this region.

(iii) Draw a clear sketch of the region whose boundary consists of portions of the x-axis, the ordinates at $x = 1$ and at $x = 2$, and the curve $y = x^2$.

(iv) Describe in geometrical terms the planar region whose points (x, y) satisfy all three inequalities

$x^2 + y^2 \le 1,\ y \le 2x,\ x \ge 0$,

and draw a sketch of this region.

5. Trigonometric Ratios — Review and Some Preliminary Results

5.1 Angles of any magnitude should be illustrated with reference to the circle $x^2 + y^2 = 1$. The sine and cosine should be defined for any angle and the other four ratios expressed in terms of these. Graphs should be drawn showing these ratios as functions of the angular measure in degrees.

5.2 The relation $\sin^2\theta + \cos^2\theta = 1$ and those derived from it should be known, as well as ratios of $-\theta$, $90° - \theta$, $180° + \theta$, $360° + \theta$ in terms of the ratios of θ. Once familiarity with the trigonometric ratios of angles of any magnitude is attained, some practice in solving simple equations of the type likely to occur in later applications should be discussed. The following examples are suitable:

$$3 + 2\cos x = 15\cos x,\ 12\sin 2x = 5\sin 36°$$

$$2\sin x = \cos x,\ 2\sin x = \tan x,\ \sec^2 x = 3$$

Later, when circular measure is introduced (in the HSC course) similar examples should be discussed again, eg, 'find a value of t for which $2 - 2\cos 2t = 0$'. Finding all the solutions in specified domains (such as 0° to 360° or $\frac{-\pi}{2}$ to $\frac{\pi}{2}$) is important in many applications. No complicated manipulation is intended.

5.3 Ratios for 0°, 30°, 45°, 60°, 90° should be known as exact values. The exercises given on this section of work should emphasise the use of the exact ratios.

5.4 The compass bearing measured clockwise from the North and given in standard three-figure notation (eg 023°) should be treated, as well as common descriptions such as 'due East', 'South–West', etc.

Angles of elevation and depression should both be defined, and their use illustrated.

5.5 The formulae

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

should be proved for any triangle. The expression for the area, $\frac{1}{2}bc\sin A$, should also be proved.

In applications of these formulae, systematic 'solution of triangles' is not required. (This is the type of exercise where the sizes of (say) two sides and one angle of a triangle are given and the sizes of all other sides and angles must be found.) The applications should be a means of fixing the results in the student's mind, and should be restricted to simple two-dimensional problems requiring only the above formulae.

Attention must be given to interpreting calculator output where obtuse angles are required.

E 5.6 3 Unit students are expected to undertake more difficult problems such as the following.

(i) The elevation of a hill at a place P due East of it is 48°, and at a place Q due South of P the elevation is 30°. If the distance from P to Q is 500 metres, find the height of the hill.

(ii) From a point A the bearings of two points B and C are found to be 333°T and 013°T respectively. From a point D, 5 km due north of A, the bearings are 301°T and 021°T respectively. By considering the triangle ABC, show that if the distance between B and C is d km, then

$$d^2 = 25\left\{\left(\frac{\sin 59°}{\sin 32°}\right)^2 + \left(\frac{\sin 21°}{\sin 8°}\right)^2 - 2\ \frac{\sin 59° \sin 21° \cos 40°}{\sin 32° \sin 8°}\right\}.$$

E 5.7 An easy, yet quite general, method of approach starts by drawing two points P and Q on the unit circle, P at an angle θ from the positive x-axis, Q at an angle ϕ. Let d be the distance from P to Q. We then compute the square of d in two ways:

(i) From the cosine rule, $d^2 = 1 + 1 - 2\cos(\theta - \phi)$.

(ii) From the cartesian coordinates of P and Q,

$$d^2 = (\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2.$$

Equating these two results, we obtain

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

We now use this basic formula to obtain all the other relations. By letting ϕ be a negative angle, say $\phi = -\psi$, and by using $\cos(-\psi) = \cos\psi$, $\sin(-\psi) = -\sin\psi$, we obtain $\cos(\theta + \psi) = \cos\theta\cos\psi - \sin\theta\sin\psi$. Next in the formula for $\cos(\theta - \phi)$, let $\theta = 90°$ and obtain $\sin\phi = \cos(90° - \phi)$. We then write $\sin(\theta + \phi)$ in the form

$$\sin(\theta + \phi) = \cos(90° - (\theta + \phi)) = \cos((90° - \theta) - \phi)$$

to derive

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Sin$(\theta - \psi)$ is obtained by the substitution $\phi = -\psi$. The sum and difference formulae for the tangent ratio are now obtained from its definition and by use of the above formulae.

The formulae for $\cos 2\theta$, $\sin 2\theta$ and $\tan 2\theta$ should be obtained explicitly as particular cases.

E 5.8 Denoting $\tan\frac{\theta}{2}$ by t, the addition formula for the tangent gives

$\tan\theta = \frac{2t}{1-t^2}$ $(t \neq \pm 1)$.

The expressions for $\cos\theta$ and $\sin\theta$ in terms of t should also be derived.

E 5.9 Sum and product formulae for the sine and cosine functions are *not* included in the 3 Unit course. The following examples illustrate the types of problem to be treated:

(i) Show that $\sin(a + b)\sin(a - b) = \sin^2 a - \sin^2 b$.

(ii) Find all angles θ for which $\sin 2\theta = \cos\theta$

(The solution is $\theta = 90° + m \times 180°$ or $30° + m \times 360°$ or $150° + m \times 360°$, where m is an arbitrary integer.)

(iii) Find all values of φ in the range of $0° \leq \varphi \leq 360°$ for which $4\cos\varphi + 3\sin\varphi = 1$

The use of the symbol $\equiv$ (*is identically equal to*) should be understood.

Note: Further practice in the solution of trigonometric equations should be given after circular measure is treated (Topic 13 of the 2 Unit syllabus).

6. Linear Functions and Lines

6.1 The linear function $y = mx + b$, for numerical values of m and b, is represented by a straight line.

The linear equation $mx + b = 0$ ($m \neq 0$) has one and only one root, which is rational if m and b are rational. This simple and apparently trivial result is worth noting; it does not extend to the quadratic equation.

6.2 The equation of a line. The geometrical significance of m and b in the linear equation $y = mx + b$.

The equation of a straight line passing through a fixed point (x_1, y_1) and having a given slope m is $y - y_1 = m(x - x_1)$.

The equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2) may be deduced from the above by noting that the slope m must be the ratio of $y_2 - y_1$ to $x_2 - x_1$, ie, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Verification that the number pairs (x_1, y_1) and (x_2, y_2) do indeed satisfy the final equation.

The linear equation $ax + by + c = 0$ represents a straight line provided at least one of a and b is nonzero.

Parallel lines have the same slope; two lines with gradients m and m' respectively are perpendicular if and only if $mm' = -1$.

6.3 If two straight lines intersect in the point (x_1, y_1), then x and y satisfy the equation for the first line and the equation for the second line; ie x_1 and y_1 satisfy two simultaneous linear equations in the two unknowns x and y.

Since we know that two lines need not intersect at all, not all pairs of linear equations can be solved simultaneously. Geometrically there is a unique point of intersection if and only if the two lines have different slopes. The cases where two lines meet, are distinct and parallel, or coincide should be related to the corresponding pairs of linear equations. For the pair $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, we have:

intersecting lines if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$;

parallel lines if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$;

coincident lines if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Any line passing through the intersection of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ has the equation

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0,$$

where k is a constant, whose value for any particular line may be found using the additional information given to specify that line.

6.4 The straight line locus $ax + by + c = 0$ divides the rest of the number plane into two regions, satisfying the inequalities $ax + by + c < 0$ and $ax + by + c > 0$ respectively. Each region is a half-plane.

Two intersecting straight lines divide the rest of the plane into four regions, each defined by a pair of linear inequalities.

(i) $a_1x + b_1y + c_1 > 0$ and $a_2x + b_2y + c_2 > 0$;

(ii) $a_1x + b_1y + c_1 > 0$ and $a_2x + b_2y + c_2 < 0$;

(iii) $a_1x + b_1y + c_1 < 0$ and $a_2x + b_2y + c_2 > 0$;

(iv) $a_1x + b_1y + c_1 < 0$ and $a_2x + b_2y + c_2 < 0$.

Each such region is the intersection of two half-planes. Extension to the case of three lines intersecting in pairs, thus including the description of the interior of a triangle as the intersection of three half-planes or the common solutions of three linear inequalities in x and y.

6.5 The formula for the distance between two points should be derived, as should the formula for the perpendicular distance of a point (x_1, y_1) from a line $ax + by + c = 0$.

For example, the following proof may be used.

The equation of the line through P, perpendicular to the given line, is

$$bx - ay = bx_1 - ay_1.$$

This meets $ax + by = -c$

at (x_0, y_0), where

$$(a^2 + b^2)x_0 = b^2x_1 - aby_1 - ac,$$

$$(a^2 + b^2)y_0 = -abx_1 + a^2y_1 - bc.$$

Thus $$x_1 - x_0 = \frac{(a^2 + b^2)x_1 - (b^2x_1 - aby_1 - ac)}{a^2 + b^2}$$

$$= \frac{a(ax_1 + by_1 + c)}{a^2 + b^2}$$

and $$y_1 - y_0 = \frac{(a^2 + b^2)y_1 - (-abx_1 - a^2y_1 - bc)}{a^2 + b^2}$$

$$= \frac{b(ax_1 + by_1 + c)}{a^2 + b^2}.$$

Hence $$(x_1 - x_0)^2 + (y_1 - y_0)^2 = \frac{(ax_1 + by_1 + c)^2}{a^2 + b^2}$$

and taking the (positive) square root gives the required distance as

$$\frac{|ax_1 + by_1 + c|}{\sqrt{(a^2 + b^2)}}.$$

E 6.6 The formula for the angle between two lines should be derived, and use made of it in solving problems.

6.7 A direct derivation of the coordinates of the midpoint of a given interval should be presented.

E The coordinates of the points dividing a given interval in the ratio $m:n$, internally and externally, should be derived.

6.8 Examples, illustrating the use of coordinate methods in solving geometrical problems, are to be restricted to problems with specified data. The following are typical problems.

(i) Show that the triangle whose vertices are (1, 1), (–1, 3) and (3, 5) is isosceles.

(ii) Show that the four points (0, 0), (2, 1), (3, –1), (1, –2) are the corners of a square.

(iii) Given that *A, B, C* are the points (–1, –2), (2, 5) and (4, 1) respectively, find *D* so that *ABCD* is a parallelogram.

(iv) Find the coordinates of the point *A* on the line $x = -3$ such that the line joining *A* to *B* (3, 5) is perpendicular to the line $2x + 5y = 12$.

7. Series and Applications

This topic might be introduced by a general discussion on series, including aspects of notation such as

$$1^2 + 2^2 + 3^2 + \ldots + N^2 = \sum_{k=1}^{N} k^2$$

There should also be justification of the topic in terms of the practical examples given below. The definitions of 'series', 'term', '*n*th term' and 'sum to *n* terms' should be understood.

7.1 The definition of an arithmetic series and its common difference should be understood. The formulae for the *n*th term and the sum to *n* terms should be derived.

7.2 The definitions of a geometric series and its common ratio should be understood, and the formulae for the *n*th term and sum to *n* terms derived.

7.3 Using a calculator, or otherwise,

$$\lim_{n \to \infty} r^n = 0 \text{ for } |r| < 1$$

should be derived. The case $|r| \geq 1$ should be discussed.

For a geometric series whose ratio r satisfies $|r| < 1$, it follows that S_n approaches a limiting value S as n increases:

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}$$

No limiting value exists for any geometric series in which $|r| \geq 1$.

NB Section 7.4 follows Section 7.5 and begins on page 46.

7.5 Applications of arithmetic series.

Applications of arithmetic series should include problems of the type 'A clerk is employed at an initial salary of $10 200 per annum. After each year of service he receives an increment of $900. What is his salary in his ninth year of service, and what will be his total earnings for the first nine years?'

Applications of geometric series should include the following types.

(i) **Superannuation**

The compound interest formula

$A_n = P(1 + r/100)^n$,

where P is the principal (initial amount), r% the rate of interest per period, and A_n the amount accumulated after n periods, should be understood.

The formula requires a calculator. The following superannuation problem can then be undertaken. 'A man invests $1000 at the beginning of each year in a superannuation fund. Assuming interest is paid at 8% per annum on the investment, how much will his investment amount to after 30 years?'

Obviously the first \$1000 is invested at 8% compound interest for 30 years, the next \$1000 for 29 years, and the last \$1000 for 1 year. Thus his investment after 30 years is (in dollars)

$1000\,(1.08^{30} + 1.08^{29} + \ldots + 1.08)$.

This is a geometric series of 30 terms, with first term 1080 and common ratio 1.08, so that this sum is

$$\frac{1080 \times (1 = 1.08^{30})}{1 - 1.08} = \frac{1080 \times 9.062657}{0.08}$$

$$= \$122\,346, \text{ to the nearest dollar.}$$

(ii) **Time Payments**

'A woman borrows \$3000 at $1\frac{1}{2}$% per month reducible interest and pays it off in equal monthly instalments. What should her instalments be in order to pay off the loan at the end of 4 years?'

Let $\$A_n$ be the amount owing after n months.

After one month and paying the first instalment $\$M$, she will owe

$3000 \times 1.015 - M = A_1$.

Similarly, $A_1 \times 1.015 - M = A_2$, and after n months,

$A_n = A_{n-1} \times 1.015 - M,$

$\quad = 3000 \times (1.015)^n - M\,(1 + 1.015 + \ldots + 1.015^{n-1})$.

But $A_{48} = 0$.

$\therefore\ M\,(1 + 1.015 + \ldots + 1.015^{47}) = 3000 \times 1.015^{48}$

$\therefore\ M = \dfrac{(1 - 1.015^{48})}{1 - 1.015} = 3000 \times 1.015^{48},$

so that

$$M = \frac{0.015 \times 3000 \times 1.015^{48}}{(1.015)^{48} - 1}$$

$$= 88.12$$

The instalment amount should be \$88.12.

Students should understand the difference between the reducible interest rate and the rate published by finance companies. The published rate in this case is the equivalent simple interest rate on \$3000 for 4 years, ie

$R = \dfrac{100I}{PT} = \dfrac{88.12 \times 48 - 3000}{3000 \times 4} \times 100 = 10.25\%$ pa.

Applications to recurring decimals

Recurring decimals should be expressed as rational numbers, eg

$$0.121212\ldots = \frac{12}{100} + \frac{12}{10^4} + \frac{12}{10^6} + \ldots$$
$$= \frac{12}{100}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right)$$
$$= 0.12 \times \frac{1}{1-0.01} = \frac{12}{99} = \frac{4}{33}$$

Other examples should also be discussed.

E 7.4 The method of proof known as 'proof by induction' makes use of a test for a set to contain the set of positive integers. This test, called the *principle of mathematical induction*, is an assumption concerning the positive integers and may be stated as follows.

'If a set of positive integers

(a) contains the positive integer 1, and

(b) can be proved to contain the positive integer $k + 1$ whenever it contains the positive integers $1, 2, \ldots, k$,

then the set contains all positive integers.'

The use of this method of proof is often suggested when a problem of the following kind arises. From given information and perhaps by experiment, we obtain a statement $S(n)$, depending on the positive integer n, which we wish to prove true for every positive integer n. We let S denote the set of positive integers n for which $S(n)$ is true. We now try to prove:

(i) that S contains 1 (ie that $S(1)$ is true), and

(ii) that if S contains $1, 2, \ldots, k$, then S contains $k + 1$
(ie if $S(1), S(2), \ldots, S(k)$ are true, then $S(k+1)$ is true).

If we manage to prove (i) and (ii), then by our test, S contains all positive integers (ie $S(n)$ is true for every positive integer n).

It frequently happens that we may be able to prove (ii) by using only the assumption that $S(k)$ is true, instead of the full assumption that $S(1), S(2), \ldots, S(k)$ are all true.

Sometimes we may guess that $S(n)$ is true only for positive integers $n \geq M$, a given positive integer. In that case we replace (i) by 'S contains M' and (ii) by 'if S contains $M, M + 1, \ldots, k$, then S contains $k + 1$'. The test enables us to conclude that S contains every positive integer greater than or equal to M.

E

Below are several applications illustrating the use of proof by induction.

1. Consider the results.

$1 = 1^2$	We call this statement $S(1)$.
$1 + 3 = 2^2$	We call this statement $S(2)$.
$1 + 3 + 5 = 3^2$	We call this statement $S(3)$.
$1 + 3 + 5 + 7 = 4^2$	We call this statement $S(4)$.

We may now guess that the following statement $S(n)$ is true for every integer n.

Statement $S(n)$: $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

The **proof** by induction of this statement consists of two steps.

Step 1. Verification that $S(1)$ is a true statement; this is easy since $S(1)$ is merely the statement $1 = 1$.

Step 2. We assume that $S(1)$, $S(2)$, …, $S(k)$ are true. We then attempt to deduce logically that $S(k + 1)$ must also be true. In the present instance, our assumption supposes that the following is true:

$S(k)$: $1 + 3 + 5 + \ldots + (2k - 1) = k^2$,

and using this we try to show that $S(k + 1)$ is true, ie, that

$1 + 3 + 5 + \ldots + (2(k + 1) - 1) = (k + 1)^2$.

By adding $2k + 1$ to each side of the (by assumption) true statement $S(k)$, we obtain

$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$,

ie

$1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$,

which is $S(k + 1)$. Thus, from the assumption that $S(k)$ is true, we have deduced that $S(k + 1)$ is true.

We have satisfied the conditions of the test for proof by induction, hence we may conclude that $S(n)$ is true for every n.

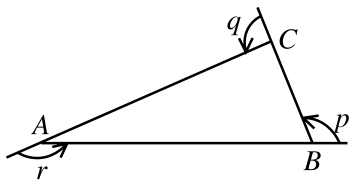
E It is important to realise that *both* steps of the method of induction must be made before the proof is valid. This can be illustrated vividly by 'proofs' of false results, for example, the 'proof by induction' that all successive integers are equal to each other (let $S(n)$ be the statement $n = n + 1$, then $S(k + 1)$ follows logically from $S(k)$, but $S(1)$ is not true).

2. The standard notation for the sum of a series should be introduced, and induction used to prove results such as:

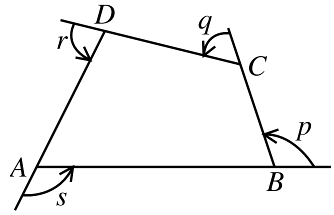
(i) $\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$

(ii) $\sum_{n=1}^{N} \frac{1}{(2n+1)(2n-1)} = \frac{N}{2N+1}$

3. If we construct a triangle ABC and measure its external angles (p, q, r in the figure) we find $p + q + r \approx 360°$.



If we construct a plane *convex* quadrilateral (for example, a parallelogram or the quadrilateral $ABCD$ shown in the figure) and measure its external angles, we again find $p + q + r + s \approx 360°$.



Generally, if we construct a plane convex polygon with n sides, and calculate the sum of the measures of its external angles, we expect the answer to be 360° for the following reason. If we were to stand at a given vertex, facing along one of its edges, and if we were to walk once around the polygon until we returned to the original vertex, facing in the original direction, then we have turned through one complete revolution (360°), and this is composed of turns of each external angle at each vertex.

E

To prove this by induction, we suppose $n \geq 3$ and let $S(n)$ be the statement 'the sum of the exterior angles of an n-sided plane convex polygon is 360°'. We now verify steps (i) and (ii).

(i) We must prove $S(3)$ is true. Referring to the figure, we must prove that $p + q + r = 360°$. Using the angle sum property of a triangle, we have

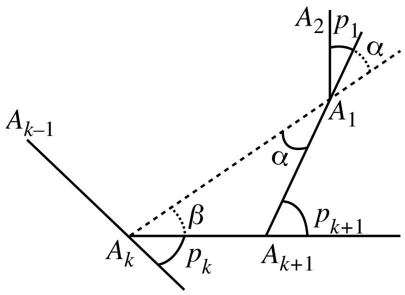
$$\hat{A} + \hat{B} + \hat{C} = 180°,$$

ie $\quad (180° - r) + (180° - p) + (180° - q) = 180°,$

or $\quad 360° = p + q + r.$

Thus $\quad S(3)$ is true.

(ii) We now suppose $S(3)$, $S(4)$, …, $S(k)$ are true, and prove $S(k + 1)$ is true. Let $A_1, A_2, \ldots, A_{k+1}$ be the vertices (in order) of a $(k + 1)$-sided plane convex polygon, with exterior angles $p_1, p_2, \ldots, p_{k+1}$ respectively. We must prove $p_1 + p_2 + \ldots + p_{k+1} = 360°$.



Join A_k to A_1 and apply $S(k)$ to the k-sided plane convex polygon $A_1A_2 \ldots A_k$. The sum of its exterior angles is therefore 360°. But

exterior angle at $A_1 = \alpha + p_1$ (see diagram),

exterior angle at $A_k = \beta + p_k$ (see diagram),

and at the other vertices $A_2, \ldots, A_{k-1}$, the exterior angle is the same as the exterior angle of the original polygon. Hence

$$(\alpha + p_1) + p_2 + \ldots + p_{k-1} + (\beta + p_k) = 360°,$$

ie $\quad p_1 + p_2 + \ldots + p_{k-1} + (\alpha + \beta) = 360°. \qquad$ (i)

But in $\Delta A_1A_kA_{k+1}$,

exterior angle at A_{k+1} = sum of interior opposite angles,

ie $\quad p_{k+1} = \alpha + \beta$ (since the angle at $A_1 = \alpha$).

Substituting this into (1) gives the result:

$$p_1 + p_2 + \ldots + p_k + p_{k+1} = 360°.$$

Thus $S(k+1)$ is true. Steps (i) and (ii) are both completed, hence we may conclude $S(n)$ is true for every $n \geq 3$.

8. The Tangent to a Curve and the Derivative of a Function

8.1 The simplest graphs studied so far have consisted almost exclusively of unbroken curves. This is a sufficient basis for the intuitive idea of continuity. The behaviour of $1/x$ and $|x|/x$ near the origin should be demonstrated, but discontinuities should not be further stressed.

8.2 Intuitively a function is 'continuous' at a given point $x = c$ if the function value $f(c)$ is 'approached continuously' from 'neighbouring' values of x, that is, if the 'limit of $f(x)$ as x approaches c' agrees with the actual function value $f(c)$ when x is precisely equal to c. Otherwise, $f(x)$ has a 'jump' at $x = c$. We use the notations

$$\lim_{x \to c} f(x), \text{ and } \lim_{h \to 0} f(c + h)$$

to mean the limit of the function as $x \to c$. If $f(x)$ is 'continuous' at $x = c$, then $\lim_{x \to c} f(x) = f(c)$, and

$\lim_{h \to 0} f(c + h) = f(c)$, for negative and positive values of h.

We use this intuitive notion to define continuity precisely as follows. A function $f(x)$ is said to be *continuous* at $x = c$ if:

(i) $f(x)$ is defined at c;

(ii) the limit of $f(x)$ as x approaches c exists;

(iii) $f(c)$ is equal to this limit.

A function $f(x)$ is called *continuous* or a *continuous function* if it is continuous at each point in its domain, ie if $f(x)$ is continuous at $x = c$ for every choice of c in the domain of the function.

There should be light treatment of the formal proofs of the limits of the sum, difference and product of two functions, and of the corollaries that, given f is continuous and g is continuous, then $f + g$, $f - g$, fg are continuous.

8.3 A secant is defined as the straight line passing through two given points on the curve. The gradients of secants for particular cases should be calculated. The general expression for the gradient of the secant through the two points $P(c, f(c))$ and $Q(x, f(x))$ on the curve $y = f(x)$ should be derived.

8.4 By drawing secants through a given point P on the curve $y = f(x)$, and through a succession of points Q_1, Q_2, Q_3, …, on the curve, first on one side of P, then on the other, the idea of the tangent line at P as the limiting position of the secant is illustrated. The geometric picture strongly suggests that there is a limiting value for the gradient of the secant through $P(c, f(c))$ and $Q(x, f(x))$ as Q approaches P, although the formula derived in 8.3 becomes meaningless if $x = c$. This limiting value of the gradient of the secant is defined to be the gradient of the tangent line (or tangent) at P and is called simply the gradient of the curve at P.

The gradient of the tangent at a specified point on a given curve should be calculated as above in a few simple cases and verified graphically.

8.5 The intuitive notion of tangent to a curve, as described above, leads to a way of defining tangent in terms of a limit. Formally, the gradient of the curve $y = f(x)$ at the point $P(c, f(c))$ is defined as the limiting value

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

provided this limit exists. Thus $f'(c)$ is the slope of the tangent line to the curve $y = f(x)$ at the point $x = c$.

The gradient of $y = f(x)$ at $x = c$ is also called the *derivative* or *differential coefficient* of $f(x)$ at $x = c$. Note that the limit may fail to exist even for a continuous curve: the curve may have a vertical tangent (infinite slope), or a sharp bend, as in $y = |x|$ at $x = 0$.

By putting $x = c + h$, where h may be positive or negative, in the definition, we get the alternative and equivalent forms

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where $\Delta x = h$, $\Delta y = f(c + h) - f(c)$, $y = f(x)$.

A number of simple numerical examples of the following type should be given in order to facilitate understanding of the above definitions.

Example. Find the derivative of the function $f(x) = x^3 + 5x$ at $x = 1$ and hence find the equation of the tangent line to the curve $y = f(x)$ at the point (1, 6).

By the definition,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1 + h)^3 + 5(1 + h) - (1 + 5)}{h} \\ &= \lim_{h \to 0} \frac{6 + 8h + 3h^2 + h^3 - 6}{h} \\ &= \lim_{h \to 0} (8 + 3h + h^2) \\ &= 8. \end{aligned}$$

The equation of the required tangent line is therefore

$y - 6 = 8(x - 1)$.

Examples should be chosen using the same function with several different (numerical) values of x, as a lead-in to the definition of the gradient function.

For the special case of the straight line $y = mx + b$, it should be verified that the definition of gradient yields the correct value m at any point $P(c, mc + b)$ on the line.

8.6 Given the function $y = f(x)$, with domain D say, we may use the definition of gradient at each point $(c, f(c))$, $c \in D$, to find out if the function has a derivative at that point. Examples should be given (eg x^2, $x^3 - x$, $|x|$) leading to the understanding that for *common* functions, a derivative exists at all points in the domain or at all points with isolated exceptions which can be identified from a sketch of the function. That $f'(c)$ is given by an expression involving c and related to the expression for $f(c)$ should also be illustrated by examples.

After first showing on a sketch of $y = f(x)$ the gradient $f'(c)$ at various points c (illustrated as the slope of the tangent to the curve at $(c, f(c))$), the transition should then be made to sketching the *gradient function* or *derivative* $y = f'(x)$, which is the function whose domain is the set of points x for which $f'(x)$ exists, and whose value at $x = c$ *is* $f'(c)$. This function should be identified and drawn for a number of functions $f(x)$ (for example, $y = f(x)$ and $y = f'(x)$ could be drawn on the same graph or one below the other, using the same line for the y-axis and different x-axes).

In adopting the usual x, y notation for the derivative function, care must be taken to avoid any confusion of notation and certainly to avoid nonsense such as

$f'(x) = \lim_{x \to x} \frac{f(x)-f(x)}{x-x}$ (ie the replacing of c by x in the definition of $f'(c)$).

Right from the beginning of work involving $f(x)$ and $f'(x)$ as functions, students must be encouraged to remember the geometrical significance of $f'(x)$ in relation to the graph of $f(x)$, and to relate properties of one graph to properties of the other. This relationship is further developed in Topic 10.

The notations $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}(f(x))$, and the use of different variables in place of x or y should be discussed and used in examples.

If $f(x)$ possesses a derivative $f'(x)$ for each x belonging to the domain of f, then $f(x)$ is called a *differentiable function*. The statement '$f(x)$ is differentiable' means '$f(x)$ has a derivative at each point of its domain'.

8.7 The geometric series $x^{n-1} + cx^{n-2} + c^2x^{n-3} + \ldots + c^{n-2}x + c^{n-1}$ has first term x^{n-1}, and ratio c/x, so that the series has the sum $(x^n - c^n)/(x-c)$. This identity and theorems on limits of sums and products may be used to find the derivative of x^n for positive integral values of n. This result should be used to find the equation of the tangent to the curve $y = x^n$ at the point (c, c^n) for positive integral values of n. Students should be asked to verify this result graphically for particular values of n and c, comparing the tangent drawn by eye with the straight line whose equation has been found analytically.

The derivative of a constant function $y = c$ is the constant function $y = 0$.

8.8 Although a few simple functions can be differentiated by straightforward use of the definition given in 8.5, ie differentiated 'from first principles', this procedure is in general far from easy. Even for such simple functions as $\sqrt{x}$ and $1/x$, we resort to the devices

(a) $$\lim_{x \to c} \frac{x^{\frac{1}{2}} - c^{\frac{1}{2}}}{x - c} = \lim_{x \to c} \frac{x^{\frac{1}{2}} - c^{\frac{1}{2}}}{(x^{\frac{1}{2}} + c^{\frac{1}{2}})\,(x^{\frac{1}{2}} - c^{\frac{1}{2}})} = \lim_{x \to c} \frac{1}{(x^{\frac{1}{2}} + c^{\frac{1}{2}})} = \frac{1}{2c^{\frac{1}{2}}}$$

$(c > 0)$;

(b) for $c \neq 0$, $\lim_{h \to 0} \left\{\left[\frac{1}{c+h} - \frac{1}{c}\right] / h\right\} =$

$$\lim_{h \to 0} \left[\frac{-h}{c(c+h)} \cdot \frac{1}{h}\right] = \frac{1}{c^2}\,.$$

Quite apart from the frequent need for ingenuity, differentiation from first principles is in general a tedious procedure.

Fortunately, there are theorems which allow us to find the derivatives of complicated functions, starting from derivatives of simpler functions. These theorems are:

if u, v are differentiable functions of x, then

(i) $\frac{d}{dx}(Cu) = C\frac{du}{dx}$ where C is a constant,

(ii) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$,

(iii) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$,

(iv) $\frac{d}{dx}F(u) = F'(u)\frac{du}{dx}$ (for any differentiable function F),

(v) $\frac{d}{dx}\left(\frac{u}{v}\right) = u\frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v}\frac{du}{dx} = \left[v\frac{du}{dx} - u\frac{dv}{dx}\right]/v^2$ $(v \neq 0)$.

While proofs of these theorems will not be examined, a satisfactory derivation is educationally desirable. In theorem (iv) it is important to give some simple examples of the meaning of the theorem before proceeding with a proof; for example use $F(u) = u^2$ and $u(x) = x^2 + 1$ to get the derivative of $(x^2 + 1)^2$, and compare with the result from the direct differentiation of $x^4 + 2x^2 + 1$.

It is definitely not the intention of this syllabus to give students a lot of drill in differentiation from first principles. On the contrary, emphasis is given to the power of general theorems, such as the ones above, to eliminate the need for extensive detailed work of this type.

8.9 It may be expedient to revise the index laws at this stage.

Use the derivative of x^n and theorems (i) and (ii) above to find the derivative of a general polynomial.

Next, put $u(x) = x^{1/n}$ and $F(u) = u^n$. Since $F(u) = x$, direct differentiation of $f(x) = F(u(x))$ gives 1, whereas the function of a function rule (theorem (iv)) leads to $nu^{n-1}u'(x)$; hence $\frac{d}{dx}\, x^{1/n} = (1/n)x^{(1/n)-1}$.

With $u(x) = x^{1/n}$ and $F(u) = u^m$ (both n and m are integers) the function of a function rule gives the derivative of a rational power m/n of x.

The derivative of the nth root of a function $f(x)$ is obtained from the function of a function rule with $u(x) = f(x)$ and $F(u) = u^{1/n}$.

It should be noted that the derivatives of $x^{1/m}$, of $x^{m/n}$, of x^{-1}, and of x^{-n}, all satisfy the general relationship $\frac{d}{dx}\, x^p = px^{p-1}$. This is not at all accidental, but rather is related to the way in which fractional powers and negative powers have been defined at an earlier stage, so as to preserve the simple index laws for more complicated indices.

Use of the rules of differentiation should be practised on functions such as $(x^2 + 1)^{\frac{1}{2}}$, $(x + 1) / (x - 1)$, $1/(x^2 - 2x + 2)$.

9. The Quadratic Polynomial and the Parabola

It is the purpose of this section to develop the algebraic properties of the quadratic function and to relate these to the parabolic curve.

9.1 **The quadratic polynomial:** $ax^2 + bx + c$ is a quadratic polynomial of the second degree or a quadratic expression. To distinguish x from the coefficients a, b, and c, it may be called an indeterminate.

When the domain of x is specified, the quadratic polynomial becomes a function. In all quadratic polynomials to be studied, the coefficients will be rational (usually integers) and the domain of x will be the set of real numbers. The quadratic function will be expressed as

$$y = ax^2 + bx + c.$$

A value of x which makes $y = 0$ is a root of the quadratic equation $ax^2 + bx + c = 0$. The term 'zero of the polynomial' might be introduced at the discretion of the teacher.

Graphs of quadratic functions: very simple examples will have already been studied. In giving further practice in graphing quadratic functions the teacher should stress, in each particular case, points of general interest, eg (1) that for large values of x the term ax^2 effectively determines the value of the function; (2) the relation between the graph and the roots of the quadratic equation $ax^2 + bx + c = 0$. Examples should include cases where the graph has respectively two points, one point, and no points in common with the x-axis.

Quadratic inequalities: the graph of the quadratic function should be used to solve quadratic inequalities, eg find the values of m for which $12 + 4m - m^2 > 0$.

9.2 Revision of simple quadratic equations which can be solved by factorisation.

Solution by 'completing the square' in particular cases. It will be noted that, applied to an equation such as $x^2 + 2x + 2 = 0$ the method leads to $(x + 1)^2 + 1 = 0$, showing that no (real) value of x can be found which will make $x^2 + 2x + 2$ equal to zero.

The traditional formula is derived by applying the method of completing the square to the general quadratic. The discriminant is to be defined and used to determine the condition for real, equal, or rational roots; pupils should be reminded of the meaning of the word 'discriminate' in ordinary language.

By actually solving the general quadratic, an important existence theorem has been established: A quadratic equation may have two (real) roots, one root or no roots. It does not have more than two roots. The relation $\alpha + \beta = -b/a$, $\alpha\beta = c/a$ between the roots α, β of a quadratic equation and its coefficients a, b, c, can be derived directly from the general solution. If an equation has roots α, β, then it is of the form

$$a(x-\alpha)\,(x-\beta) = 0 \text{ or } a[x^2 - (\alpha + \beta)x + \alpha\beta] = 0.$$

Exercises involving finding equations whose roots bear stated relations to the roots of some other equation are not included in this syllabus.

9.3 From $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$

(the general result would of course be preceded by particular examples), the conditions for positive definite, negative definite and indefinite quadratic expressions are derived. Only if $b^2 > 4ac$ can the expression take both positive and negative values, and it has the same sign as a for all values of x except those lying between the roots of the equation $ax^2 + bx + c = 0$. Also, $ax^2 + bx + c$ has its greatest or least value when $x = -b/2a$ and this greatest or least value is $(4ac - b^2)/4a$.

An alternative treatment is to consider the roots of the expression

$$f = ax^2 + bx + c \ (a \neq 0).$$

(i) Suppose the discriminant $\Delta = b^2 - 4ac < 0$. Then f cannot be zero. Thus if $\Delta < 0$ and $a > 0$, then $f > 0$ for all values of x, and is called positive definite. If $\Delta < 0$ and $a < 0$, then $f < 0$ for all values of x, and is called negative definite.

(ii) If $\Delta > 0$, then $f = 0$ for two distinct values of x, say x_1 and x_2. The greatest or least value of f occurs at $x = \frac{1}{2}(x_1 + x_2)$ and f takes both positive and negative values.

(iii) If $\Delta = 0$, then $f = 0$ for *one* value of x, at $x = -b/2a$. Then $f \geq 0$ if $a > 0$, and $f \leq 0$ if $a < 0$, for all values of x.

(iv) The turning point of f is at $\frac{df}{dx} = 0$, ie at $x = -b/2a$.

Students should learn to find the turning point and zeros (if any) of f in order to sketch the graph of f.

The identity of two quadratic expressions

Theorem: If $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$ for more than two values of x, then

$$a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2.$$

The proof reduces to a discussion of the equation $ax^2 + bx + c = 0$ with $a = a_1 - a_2$, $b = b_1 - b_2$, and $c = c_1 - c_2$. Beginners find the proof elusive. Work on quadratic equations has shown that $ax^2 + bx + c = 0$ can vanish for *at most* two values of x. There is one exception: if $a = b = c = 0$, the expression vanishes for all values of x. If it is given that $ax^2 + bx + c = 0$ for more than two values of x, we must conclude that $a = b = c = 0$. Otherwise the data presents us with a contradiction.

Examples should include the expression of a quadratic polynomial $ax^2 + bx + c$ in the form $Ax(x-1) + Bx + C$, where $C = c$, $A = a$, $B = a + b$, the fitting of a quadratic to three given function values, and similar identities.

9.4 Examples of the following kinds should be discussed:

$$x^4 - 4x^2 - 12 = 0,$$

$$(x + 1)^2 = 4x^2,$$

$$9^x - 4(3)^x + 3 = 0,$$

E $$(x + \frac{1}{x})^2 - 5(x + \frac{1}{x}) + 6 = 0.$$

9.5 A parabola is defined as the locus of a point which moves so that its distance from a fixed point is equal to its distance from a fixed line.

If the fixed point is $(0, A)$ and the fixed line is $y = -A$, the equation of the locus is

$$x^2 = 4Ay \text{ or } y = x^2/4A.$$

Definitions of *focus*, *directrix*, *vertex*, *axis* and *focal length* should be given and illustrated by examples.

By considering, for example, cases where:

(a) the focus is (x_0, A) and the directrix is $y = -A$

(b) the focus is $(0, y_0 + A)$ and the directrix is $y = y_0 - A$

(c) the focus is $(x_0, y_0 + A)$ and the directrix is $y = y_0 - A$

the interpretation of the equation

$$(x - x_0)^2 = 4A(y - y_0)$$

as representing a parabola with vertex (x_0, y_0), axis $x = x_0$, focus $(x_0, y_0 + A)$ and directrix $y = y_0 – A$ should be treated. Similarly, the equations $(x – x_0)^2 = –4A(y – y_0)$, $(y – y_0)^2 = 4A(x – x_0)$, $(y – y_0)^2 = –4A(x – x_0)$ should be discussed.

Starting with the general quadratic function

$$y = ax^2 + bx + c \ (a \neq 0),$$

and rewriting it as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right)$$

shows that its graph is a parabola whose focus, directrix, vertex and axis are easily found.

Practice should also be given in finding the equation of a parabola given, for example, its vertex, axis and focal length, or in finding the equation of the family of parabolas having, for example, the line $x = x_0$ as axis and a given vertex or focal length or passing through a given point.

E 9.6 The parametric equations $x = 2At,\ y = At^2$.

The equations of the tangent and the normal to the parabola at the point *‘t’* and at the point (x_1, y_1).

The equation of the chord of contact of the tangents from an external point.

The following geometrical properties of the parabola should be proved analytically.

(i) The tangent to a parabola at a given point is equally inclined to the axis and the focal chord through the point. The significance of this result in the principle of the parabolic reflector should be mentioned.

(ii) The tangents at the extremities of a focal chord intersect at right angles on the directrix.

Simple locus problems; the following is typical of the most difficult problems to be treated.

The normals to the parabola $x^2 = 4Ay$ at the points P_1 and P_2 intersect at Q. If the chord P_1P_2 varies in such a way that it always passes through the point $(0, –2A)$, show that Q lies on the parabola.

10. Geometrical Applications of Differentiation

This is a continuation of the exploration of the relationship between geometrical properties of functions and analytic properties of functions begun in Topic 4 and developed further in Topic 8. In particular, it will be found useful to consider examples in which the graphs of $y = f(x)$ and $y = f'(x)$ are drawn so that visual transfer of information occurs easily. In addition, the interpretation of $f'(x)$ as the gradient of the tangent at $(x, f(x))$ is usefully retained by drawing tangent lines to $y = f(x)$ at appropriately chosen values of x.

10.1 The geometrical significance of the sign of $f'(x)$ is to be understood, including the determination of whether or not $f(x)$ is increasing or decreasing.

10.2 A *stationary point* of $f(x)$ is defined to be a point on $y = f(x)$ where the tangent is parallel to the x–axis.

At such a point, $\frac{dy}{dx} = 0$.

A *turning point* of $f(x)$ is a point where the curve $y = f(x)$ is locally a maximum or a minimum. For differentiable functions f, all turning points are stationary; but there are stationary points of some functions at which the tangent 'crosses the curve' and which are not turning points. (The term *inflexion* might be deferred for the present to avoid the mistaken idea that all inflexions are 'horizontal'.) Thus the criterion for a turning point is the change in sign of $f'(x)$ as x passes through the abscissa of the point, and the identification of type of stationary point should be made by considering the sign of $f'(x)$ on either side of the point.

The distinction between a local maximum and the greatest value of a function for a given domain of the variable should be made clear.

10.3 The definition of the second derivative and the notations

$$f''(x),\ \frac{d^2y}{dx^2},\ y''.$$

10.4 Geometrical significance of the sign of the second derivative:

if $\frac{d^2y}{dx^2} > 0$ at P, the curve is concave upwards at P;

if $\frac{d^2y}{dx^2} < 0$ at P, the curve is concave downwards at P.

At a point of inflexion, $\frac{d^2y}{dx^2}$ vanishes and its sign changes on passing through the point.

The second derivative may be used to distinguish between maximum and minimum turning points. The criterion should be used with caution, since the condition

$$\frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} \neq 0,$$

is sufficient but not necessary. For example, $y = x^4$ has a minimum point at the origin where

$$\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0.$$

On the other hand, the curve $y = x(3x - 3 - x^2)$ has a point of inflexion at $x = 1$.

10.5 The sketching of curves such as quadratics, cubics and higher polynomials and simple rational functions. After computing some values (which may include points where $x = 0$ and where $y = 0$), the determination of the stationary points is frequently very useful. Other considerations are symmetry about the axes, behaviour for very large positive and negative values of x, and the points at which functions such as:

$$y = \frac{1}{x-1},\ y = x + \frac{1}{x},\ y = \frac{1}{x}$$

are defined.

E | Examples should include both horizontal and vertical asymptotes, eg $y = \frac{x}{x^2 + 1}$, $y = \frac{x - 1}{x^2 - 9}$.

For all students the techniques developed in this topic should be applied to examples involving functions introduced later in their courses.

10.6 Problems on maxima and minima should include the identification of turning points for curves, the finding of maximum and minimum values of given functions over different intervals and over their domains, and the treatment of problems for which the appropriate function to be analysed is to be constructed from data given in words or on a diagram.

10.7 The equations of tangents and normals to curves should be found for simple curves. Curves in which the differentiation involves heavy mechanical work should be avoided.

10.8 Given $f'(x)$, the question 'What is $f(x)$?' naturally arises. Particular examples will make plausible, and a proof will show, that $f(x)$ is not uniquely determined but that two functions which have the same derivative can only differ by a constant.

Geometrically, given the gradient function of a curve, the curve is not fixed but it is one of a 'family' of similar curves. For example, if

$$\frac{dy}{dx} = 2x,$$

then

$$y = x^2 + c$$

and for different values of c a family of parabolas is obtained.

The term integration need not be used at this stage. It is in fact preferable to avoid it. Primitive function is a correct and suitable term.

The following are typical examples.

(i) Given $\frac{d^2s}{dt^2} = 2$, find s in terms of t if $\frac{ds}{dt} = 10$ and $s = 0$ when $t = 0$.

(ii) The gradient function of a curve is $3x^2 - 1$ and the curve passes through the point (4, 1). Find its equation.

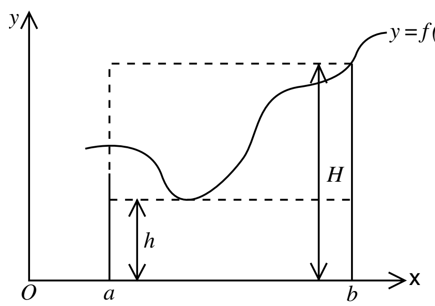
11. Integration

11.1 The notion of limit underlies both the differential calculus and the integral calculus. In the former, the intuitive geometrical notion of a tangent to a curve is formalised by the definition of gradient; if the gradient $f'(x)$ exists at $(x, f(x))$, the slope of the tangent at the point is defined to be $f'(x)$. The intuitive geometrical notion of area under a curve provides the basis for the development of the integral calculus. Its formal development requires that this area be defined as the limit (if it exists) of a certain sum of approximating rectangles, but this development is complicated by two factors which do not arise in the case of the gradient. Firstly, the appropriate limit is more difficult to specify, and secondly, having specified the limit, the specification usually does not allow the value of the limit to be calculated easily.

For these reasons, the development of this topic will not follow that used for the differential calculus. Although the ideas leading to a formal definition of an area under a curve are to be discussed and illustrated with examples, the formal definition itself is not required. It is to be assumed that for common functions (and certainly for all continuous functions) there is an analytical formulation, in terms of a limit operation, of the intuitive idea of an area under a curve resulting in the *definite integral* becoming defined, when it exists, as the measure of this area.

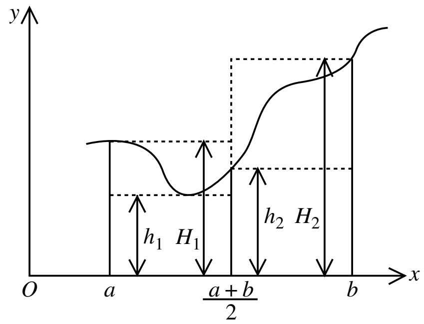
Fortunately, the discovery that the differential calculus and the integral calculus are related provided a much simpler method of evaluating definite integrals of most of the common functions. This discovery (the '*fundamental theorem of the calculus*') can be presented in a simple manner (provided the existence of area is assumed) and forms the basis of much theoretical and practical work involving integration.

We complete this section by briefly describing a suggested method of introducing the topic. We suppose $y = f(x)$ is defined for values of x including all x in $a \leq x \leq b$, is positive for these values of x, and has an easily drawn graph:



Intuitively, there is an area enclosed by the x-axis, the ordinates at $x = a$ and $x = b$, and the curve. The problem is to calculate the size A, say, of this area. We know how to calculate areas of rectangles, triangles and of simple polygons. If h, H are respectively the minimum and maximum values of $f(x)$ in $a \le x \le b$ (and examples should be chosen so that these occur both inside and at the endpoints), then

$$h(b - a) \le A \le H(b - a).$$



If we now split $a \le x \le b$ into two subintervals, $a \le x \le \frac{a+b}{2}$ and $\frac{a+b}{2} \le x \le b$, take minimum and maximum values h_1, H_1 in the first subinterval and h_2, H_2 in the second, then the diagram shows that

$$h(b - a) \le (h_1 + h_2)\,\frac{b-a}{2} \le A \le (H_1 + H_2)\,\frac{b-a}{2} \le H(b - a),$$

and usually the inequalities are strict, so that the new area sums are closer to A than the original bounds.

Taking three, four, … , n equal subdivisions of $a \le x \le b$ and forming the corresponding area sums, gives closer and closer bounds for A. This is to be done for a simple function such as $y = x^2$ for $0 \le x \le 1$, the sums to be written down and evaluated using a calculator.

Intuitively, as the number n of subdivisions increases, the approximating sums approach the value A. Supposing n large, a typical rectangle in such a sum has a small base of length dx ('dx' or 'Δx' are notations used for a very small length) and a height which is close to $f(x)$ for *any* value of x lying in its base. This is so because *f continuous* means that all values $f(x)$ are close together if the values x are close together. Thus the area of a typical rectangle is $f(x)dx$ and the sum of these areas is represented symbolically by $\sum f(x)dx$. The limiting value of this sum as n increases (and dx decreases) was denoted symbolically by

$$\underset{x=a}{\overset{x=b}{S}} f(x)dx,$$

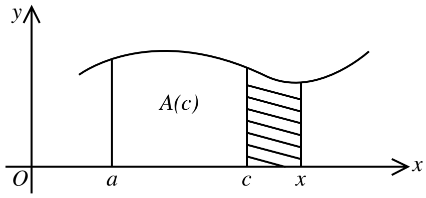
where the large S stood for 'limiting sum' and the bounds $x = a$ and $x = b$ indicated the interval over which the sum was taken. As time went by the S became elongated and modern notation for the same limit is

$$\int_a^b f(x)dx,$$

which is called the *definite integral of the function* $f(x)$ *between* $x = a$ *and* $x = b$ and whose value A is the size of the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

11.2 Recall that a *primitive function* of a given function $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$. Any two primitive functions of $f(x)$ differ by a constant function.

Let $y = f(x)$ be a continuous positive curve defined for values of x including all $x \geq a$.



If $c > a$, let $A(c)$ denote the area between a and c, and $A(x)$ denote the area between a and x. Then $A(x) - A(c)$ denotes the shaded area, which is approximately that of a rectangle of height $f(x)$ and base $x - c$, and so of area $f(x)\,(x - c)$. Thus the ratio

$$\frac{A(x) - A(c)}{x - c}$$

is approximately equal to $f(x)$. If the maximum and minimum values of $f(t)$ for $c \leq t \leq x$ are M, m respectively, then precise bounds for this ratio are

$$m \leq \frac{A(x) - A(c)}{x - c} \leq M,$$

and these inequalities also hold if $x < c$.

As x approaches c, both m and M approach $f(c)$. Hence

$$\lim_{x \to c} \frac{A(x) - A(c)}{x - c} = f(c)$$

and (by the definition of the derivative) is also equal to $A'(c)$. Hence

$$A'(c) = f(c)\ (c > a),$$

and this equation, being true for all $c > a$, says that the two functions $y = f(x)$ and $y = A'(x)$ are equal for all $x > a$. Hence $A(x)$ is a primitive function of $f(x)$ for $x > a$. If we know already a primitive function $F(x)$ of $f(x)$, then

$$A(x) = F(x) + C \text{ for some constant } C.$$

As x approaches a, $A(x)$ approaches 0, $F(x)$ approaches $F(a)$ and consequently the value of C is $-F(a)$. Thus

$$A(x) = F(x) - F(a)\ (x > a).$$

If $b > a$, then

$$A(b) = \int_a^b f(x)dx = F(b) - F(a),$$

and hence the area and the definite integral can both be found using the known primitive function $F(x)$. The problem of evaluating definite integrals is therefore solved if we can find a primitive function. For example, if $f(x) = x^2$, we may choose $F(x) = \frac{x^3}{3}$, and

$$\int_a^b x^2 dx = F(b) - F(a) = \tfrac{1}{3}b^3 - \tfrac{1}{3}a^3.$$

This is an appropriate time to introduce and discuss the following extensions.

(i) The case where $f(x)$ is negative or changes sign.

For example, if $f(x) = -x^3$, draw $y = f(x)$ for $x \geq -1$ and evaluate, say,

$$\int_{-1}^{-\frac{1}{2}} -x^3 dx,\ \int_{-1}^{0} -x^3 dx,\ \int_{-1}^{-\frac{1}{2}} -x^3 dx,\ \int_{-1}^{1} -x^3 dx,$$

$$\int_{-1}^{2} -x^3 dx \text{ and } \int_{0}^{1} -x^3 dx,\ \int_{0}^{2} -x^3 dx,$$

using the primitive function $\frac{-x^4}{4}$. From this, the idea that area is evaluated as positive or negative according as f is positive or negative over the interval, and also that when f changes sign the integral gives the net area (positive + negative) should be developed.

(ii) The result that if $a \leq b \leq c$,

$$\int_a^b f(x)dx + \int_b^c f(x)dx - \int_a^c f(x)dx,$$

obtained by writing down values in terms of a primitive $F(x)$, should be related to equations involving the corresponding signed areas.

(iii) If F, G are primitives of f, g respectively, then $F + G$ is a primitive of $f + g$. Thus

$$\begin{aligned}\int_a^b (f(x) + g(x))dx &= (F(b) + G(b)) - (F(a) + G(a))\\ &= (F(b) - F(a)) + (G(b) - G(a))\\ &= \int_a^b f(x)dx + \int_a^b g(x)dx.\end{aligned}$$

The practical importance of this result is to be emphasised; the integral of any sum is a sum of the integrals, so any polynomial may be integrated by finding primitives of each term, etc.

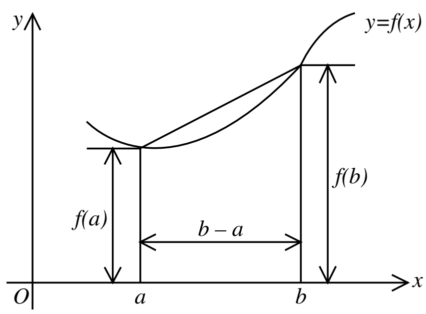
There is no need to discuss the case $\int_a^b f(x)dx$, where $b < a$, immediately, but it should be introduced at an appropriate time, with simple examples.

11.3 There are quite simple functions whose definite integrals cannot be found exactly in terms of common functions. For such cases, for problems involving more complex functions, and also because of the ease with which computers can execute the necessary calculations to a high degree of accuracy, this section is devoted to two of the simple methods of approximate integration based on the idea of approximating an exact area by a sum of areas of shapes whose areas can be calculated. As well as using simple polygons such as rectangles or trapezia, one may use any curve $y = f(x)$ whose area can be calculated explicitly; for example, quadratic polynomials (parabolas) or cubic polynomials.

Examples should be confined to problems in which the resulting calculations are easily performed on a calculator, but should include functions whose integrals are discussed or occur later in this syllabus when examples involving them may appropriately be introduced.

(i) **Trapezoidal rule**

If we approximate the curve $y = f(x)$ on the interval $a \le x \le b$ by the straight line passing through $(a, f(a))$ and $(b, f(b))$, we may estimate the area under the curve by the area under the line:

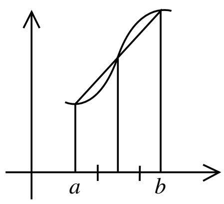


This approximates $\int_a^b f(x)dx$ by the area of a trapezium, which may be calculated as $\frac{b-a}{2}(f(a) + f(b))$.

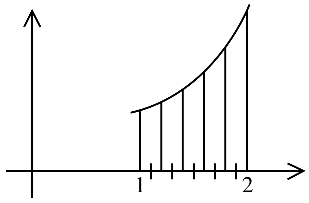
If $f(x) = x^2$, $a = 1$ and $b = 2$, $\int_1^2 x^2 dx = \frac{7}{3} = 2.33\ldots$,
while the above approximation gives $\frac{1}{2}(f(1) + f(2)) = \frac{5}{2} = 2.5$.
By dividing the base into two equal subintervals, and using a linear approximation to $f(x)$ on each subinterval, the area is approximated by two trapezia of total area

$$\frac{b-a}{4}(f(a) + 2f\frac{a+b}{2} + f(b)),$$

which for the given example yields the better approximation $(1 + 9/2 + 4) / 4 = 19/8 = 2.375$.



The method extends to dividing the interval into n equal subintervals of length $h = (b - a)/n$, and using a trapezium approximation on each subinterval. For instance, using $n = 5$ (so $h = 0.2$) in our example



$$\int_1^2 x^2 dx \approx \frac{0.2}{2}\ [1 + 2(1.44) + 2(1.96) + 2(2.56) + 2(3.24) + 4]$$
$$= 2.34.$$

(ii) **Simpson's rule**

If we approximate the curve $y = f(x)$ on $a \le x \le b$ by a quadratic function (ie by a parabola) agreeing with $f(x)$ at the three points $(a, f(a))$, $\left(\frac{a+b}{2},\ f\left(\frac{a+b}{2}\right)\right)$ and $(b, f(b))$, then, as a diagram will show, we would expect to increase the accuracy of the area approximation by using the area under the parabola. The resulting approximation, known as Simpson's rule, is

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$$

Applied to the same integral as before, we obtain

$$\int_1^2 x^2 dx \approx \frac{1}{6}\ \{1 + 9 + 4\} = \frac{7}{3}\ ,$$

which is of course exact because the approximating parabola is in this case the same function $y = x^2$. (In fact, Simpson's rule is *exact* for $f(x)$ equal to any quadratic or cubic polynomial.)

An example such as $y = \frac{1}{x}$ for $1 \le x \le 2$ should now be chosen, where students cannot find a primitive function, and the integral $\int_1^2 \frac{1}{x}dx$ approximated by the trapezoidal rule with $n = 1$ and $n = 2$, and by Simpson's rule applied first to the whole interval and then to the two subintervals $1 \le x \le \frac{3}{2}$ and $\frac{3}{2} \le x \le 2$.

It is important to emphasise that any definite integral may be evaluated approximately by such rules, and that estimates of the size of possible error can often be calculated. Later, when integrals are used to find volumes or velocities, etc, examples should be chosen to obtain approximate values of the relevant quantities.

11.4 (i) **Areas**

Only very simple functions can be considered at this stage. Examples should include the calculation of areas bounded by a curve and the x-axis, by a curve and the y-axis, and between two curves. Simple examples where the function changes sign in the interval of integration should also be included. All problems should be accompanied be a clear sketch showing the required area.

(ii) **Volumes of solids of revolution**

Standard results for the cone and the sphere should be derived. Other problems should involve only simple curves, but revolution about the y-axis should also be considered.

(iii) Other applications occur in later topics, and care should be taken to ensure that examples and problems in the later topics are used to reinforce integration techniques.

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11.5 3 Unit students should be prepared to undertake some harder integration, as exemplified below.

(i) Find $\int x\sqrt{(1 + x^2)}dx$, using the substitution $u = x^2 + 1$.

(ii) Use the substitution $t = u^2 - 1$ to evaluate

$$\int_0^1 \frac{t}{\sqrt{(1+t)}}\,dt.$$

In all examples the substitution is to be given.

12. Logarithmic and Exponential Functions

Note: This particular section might well benefit by splitting it into two parts. The algebraic Topics 12.1–12.3 can be taught before calculus is introduced. Keys for the exponential and logarithmic functions appear on most scientific calculators and students should be familiar with their use. Students should be aware of the existence of tables of these functions, and should check that the values given in the tables are those appearing on the calculator.

12.1 The introduction of scientific calculators into the classroom in the junior school eliminates the necessity of practical work with logarithms to the base of 10. Even though familiarity with index notation is expected of the users of such calculators, it may be necessary to revise or to develop that notation and the index laws for integer exponents. The relationship of indices to multiplication of repeated factors and the introduction of zero and negative integer indices should be understood, as should the fact that the index laws (obtained initially for positive integer indices) remain valid for arbitrary integer indices.

If $a > 0$ and $r = \frac{p}{q} > 0$, then a^r is defined as the qth root of a^p. This should be approached via simple values of r (eg $\frac{1}{2}$, $\frac{2}{3}$).

The index laws should be verified for simple cases (such as $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^1$) and stated to be true for general rational indices (a general proof is not required). The extension to negative rational indices follows as for the case of integer indices.

Calculators should be used to verify numerical examples of index notation and index laws. It is expected that algebraic computations involving the index rules will be tested in the examination.

12.2 The words *exponent*, *base* and *logarithm* should be introduced and understood. The notations $\log_a x$, $\log_{10} x$ should be known. There should be some examples of cases where the logarithm is integral and rational, and other examples using calculator or tables.

The algebraic properties of logarithms and exponents (including in particular, the identities $\log_a 1 = 0$, $\log_a xy = \log_a x + \log_a y$, and $\log_a x^c = c \log_a x$) should be derived from the appropriate index laws.

12.3 For a fixed $a > 0$, calculators should be used to draw graphs of $y = a^x$ and $y = \log_a x$, the cases $a < 1$, $a = 1$, $a > 1$ to be discussed. (Some discussion could be introduced here, or into 12.2, regarding the problem of defining a^x when x is real but not rational, and it could be pointed out that the method used in 12.1 to define, say, $a^{\frac{1}{2}}$ simply does not extend to cover the case of attaching a meaning to say $a^{\sqrt{2}}$ or a^{π}. The idea of using rational approximations to real exponents (rational and non-rational) could be explored using calculators.)

If $a > 1$, it should be stated that $y = a^x$ is an increasing function which takes each positive real value once only. Thus to each positive y there is a unique real x such that $y = a^x$ and hence such that $\log_a y = x$. The change of base formulae follow from index laws. Computational examples of the algebraic properties of logarithms and change of base should be set, and may be tested in the examination.

12.4 (a) To find the gradient of the curve $y = 10^x$.

(i) Let $f(x) = 10^x$. At any value x, we know that

$$f'(x) = \lim_{h \to 0} \left[\frac{10^{x+h} - 10^x}{h}\right] = 10^x \lim_{h \to 0} \left(\frac{10^h - 1}{h}\right)$$

(ii) Evaluate $\left(\frac{10^h - 1}{h}\right)$ for various values of h, using a calculator,

eg $h = 0.1 \quad 0.01 \quad 0.001$

$\left(\frac{10^h - 1}{h}\right) = 2.6 \quad 2.33 \quad 2.31.$

It appears that $\left(\frac{10^h - 1}{h}\right)$ approaches a limit (call it γ). This is also intuitively obvious, since $\lim_{h \to 0}\left(\frac{10^h - 1}{h}\right)$ is the gradient of the curve $y = 10^x$ at $x = 0$.

(b) Let $\gamma = \lim_{h \to 0}\left(\frac{10^h - 1}{h}\right) \approx 2.3$. If $y = 10^x$, then by (a) (i), $\frac{dy}{dx} = \gamma 10^x$.

If $z = 10^{\lambda x} = y^{\lambda}$, then $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = \lambda y^{\lambda - 1}\, \gamma 10^x = \lambda\gamma 10^{\lambda x}$.

Now choose λ so that $\lambda\gamma = 1$ (ie $\lambda \approx 0.43$) and write e $= 10^{\lambda} \approx 2.7$.

Then $z = e^x$ and $\frac{dz}{dx} = e^x$.

(If calculators are used, the argument may be applied to a^x for other a, such as $a = 2$.)

(c) Alternative methods of calculating e are:

(i) put $H = 10^h - 1$ so that $h = \log_{10}(1 + H)$ and $H \to 0$ as $h \to 0$. Then

$$\frac{1}{\gamma} = \lim_{H \to 0}\left(\frac{\log_{10}(1 + H)}{H}\right);$$

(ii) put $\frac{1}{n} = 10^h - 1$ so that $h = \log_{10}(1 + \frac{1}{n})$ and $n \to \infty$ as $h \to 0$. Since

$$\frac{h}{10^h - 1} = n \log_{10}(1 + \tfrac{1}{n}) = \log_{10}(1 + \tfrac{1}{n})^n,$$

it follows from (b) that $e = 10^{1/\gamma} = \lim_{n \to \infty} (1 + \frac{1}{n})^n$.

(d) If $y = e^x$, $\frac{dy}{dx} = e^x = y$, so $\frac{dx}{dy} = \frac{1}{y}$.

$\therefore\ x = \log_e y = \int \frac{1}{y}\, dy$, etc.

(e) Teachers may prefer to use for able students the approach, based on $\int_1^x \frac{dt}{t}$, which first defines ln x and then defines e^x as the function inverse to ln x.

The notations ln x and log x for the natural logarithm $\log_e x$ should be known. For historical reasons some books and calculators use log x for $\log_{10} x$. This practice should be discouraged.

12.5 (i) Differentiation of e^{ax+b}, $\log_e(ax + b)$ and the corresponding integrations.

(ii) Differentiation of $\log_e f(x)$ for simple functions $f(x)$.

(iii) Integration of $f'(x)/f(x)$ by inspection, without formal change of variable.

13. The Trigonometric Functions

13.1 Just as natural logarithms are preferable to logarithms to other bases (including 10) for the study of logarithmic and exponential functions, it turns out that degrees are not a satisfactory measure of angle size in further work involving the trigonometric functions. A suitable measure of angle size is the length of arc subtended by the angle when it is at the centre of a unit circle. An angle is of measure 1 if the arc it subtends on a unit circle has unit length. This new unit of measure is called a *radian*.

The number π may be defined as the length of a semi-circular arc of a unit circle. It then follows that

π radians = 180°,

and the formulae for conversion from degrees to radians and vice versa now follow. Familiarity with both measures is expected. Practice should be given so that exact equivalents are known for common angle sizes and so that accuracy is developed in approximating sizes given in one measure by sizes in the other. The formula $\ell = r\theta$, for the length ℓ of an arc subtending an angle at the centre of a circle of radius r, should be derived, as also should the formula $A = \frac{1}{2} r^2\theta$ for the area of the corresponding sector.

The relations treated in Topic 5.2 should be revised using radian measure, as should Topic 5.9 for 3 Unit students.

13.2 Using radian measure, sine and cosine are now defined as functions of a *real* variable: for each real number x, $\sin x$ is defined as the sine of an angle of size x radians, $\cos x$ as the cosine of this angle. Thus the functions $y = \sin x$, $y = \cos x$ are defined for all real x and graphs should be drawn of them. The functions $\tan x$ etc, may now be defined in terms of $\sin x$ and $\cos x$, their domains of definition are to be found, and graphs drawn of them.

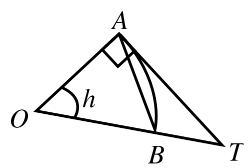
13.3 The graphs of $\sin x$, $\cos x$ and $\tan x$ should be known and their periodicity noted. Graphs of functions such as $y = 3 \cos 2x$, $y = \sin \pi x$ or $y = 1 - \cos x$ should be drawn, and the main features noted.

Some practice is to be given in using graphs to solve simple equations such as

$\sin 2x = \frac{1}{2} x$.

13.4 and 13.5 It should be noted that $\sin h \to 0$ and $\cos h \to 1$ as $h \to 0$.

The limit $\frac{\sin h}{h} \to 1$ as $h \to 0$, should be obtained. This can be tentatively derived on a calculator, or the following geometrical proof can be used.



AB is an arc of the unit circle, h is in radians. *O* is the centre of the circle and *AT* is the tangent at *A*.

Area ΔOAB < area sector OAB < area ΔOAT, ie,

$$\tfrac{1}{2}\sin h < \tfrac{1}{2}h < \tfrac{1}{2}\tan h.$$

$$\therefore\ 1 < \frac{h}{\sin h} < \frac{1}{\cos h}.$$

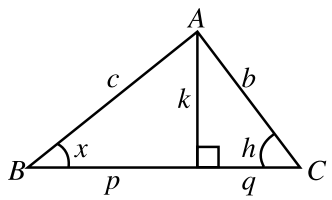
As $h \to 0$, $\cos h \to 1$. Hence $\dfrac{h}{\sin h} \to 1$ as $h \to 0$.

This procedure can be repeated with h in degrees to illustrate the reason for introducing radians. Moreover,

$$\frac{\sin^2 h}{h^2} = \frac{1-\cos^2 h}{h^2} = \frac{(1-\cos h)(1+\cos h)}{h^2}.$$

Thus $\dfrac{1-\cos h}{h^2} = \dfrac{1}{1+\cos h}\left(\dfrac{\sin h}{h}\right)^2 \to \tfrac{1}{2}$ as $h \to 0$.

The evaluation of $\dfrac{d}{dx}(\sin x)$ given below uses these limits and the formula for $\sin(x+h)$. A simple derivation of this formula, valid for $0 < x < \pi$ and for small positive values of h is as follows.



In ΔABC,

$$\text{Area } ABC = \tfrac{1}{2}cb\sin A;$$
$$= \tfrac{1}{2}cb\sin(x+h);$$
$$= \tfrac{1}{2}kp + \tfrac{1}{2}kq.$$

Hence $\sin(x+h) = \dfrac{k}{b}\cdot\dfrac{p}{c} + \dfrac{k}{c}\cdot\dfrac{q}{b}$,

$= \sin h \cos x + \sin x \cos h.$

Proofs of results in 13.5 are not examinable and alternative methods of proof may be used at the teacher's discretion.

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3 Unit students should be able to use the result $\displaystyle\lim_{h\to 0}\frac{\sin h}{h} = 1$.

Example: given $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$, find $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{2x}$

The derivative of $\sin x$.

$$\frac{\sin(x+h) - \sin x}{h} = \left(\frac{\sin h}{h}\right)\cos x + \left(\frac{\cos h - 1}{h}\right)\sin x.$$

As $h \to 0$, $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to -\dfrac{h}{2} \to 0$.

Hence $\dfrac{d}{dx}\sin x = \displaystyle\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x$.

Since $\cos x = \sin(\pi/2 - x)$, the function of a function rule gives

$$\frac{d}{dx}\cos x = -\cos(\pi/2 - x),$$
$$= -\sin x,$$

and finally

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}(\sin x/\cos x) = \sec^2 x.$$

E | 13.6 3 Unit students should be able to perform integrations of the types

$$\int 2\cos^2 x\, dx, \int_0^{\frac{\pi}{4}} \cos x \sin^2 x\, dx.$$

13.7 $\frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$, hence

$$\int \cos(ax + b)dx = \frac{1}{a}\sin(ax + b)\ (a \neq 0).$$

Other pairs should be derived similarly. Examples giving practice in differentiation and integration of these functions and related to items discussed in topics 10 and 11 should also be given.

14. Applications of Calculus to the Physical World

14.1 The rate of change of some physical quantity Q is defined as $\frac{dQ}{dt}$.
(An alternative notation is $\dot{Q}$). This can be justified by considering $\lim_{s \to t} \frac{Q(s) - Q(t)}{s - t}$. The rate of change of the population P of a town is defined, for example, as $\frac{dP}{dt}$ while the rate of change of the volume of water in a container would be defined as $\frac{dV}{dt}$.

In doing questions on rates of change students should be encouraged to draw sketches of Q and $\dot{Q}$ as functions of t whenever this is possible. In particular the relationship between an integral and the area under a curve is relevant here.

Examples should be kept as mathematically simple as possible, with the emphasis on understanding the behaviour of the system. 2 Unit students will not be expected to derive an equation from given information; in all examples, an equation will be given.

Examples

(i) A valve is slowly opened in a pipeline such that the volume flow rate R varies with time according to the relation

$R = kt\ (t > 0)$, where k is a constant.

Calculate the total volume of water that flows through the valve in the first 10 seconds if $k = 1.3\text{m}^3\text{s}^{-2}$.

$R = \frac{dV}{dt} = kt.$

$\therefore\ V = \frac{1}{2}kt^2 + c$. Since $V = 0$ at $t = 0$, $c = 0$.

When $t = 10$, $V = \frac{1}{2} \times 1.3 \times 10^2 = 65\ \text{m}^3$.

65 m^3 flows through in the first ten seconds.

The following are examples of harder problems.

E (ii) A spherical balloon is being deflated so that the radius decreases at a constant rate of 10 mm per second. Calculate the rate of change of volume when the radius of the balloon is 100 mm.

Let V be the volume of the balloon. Then $V = \frac{4}{3}\pi R^3$, and we are given that $\frac{dR}{dt} = -10$. Using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dR} \cdot \frac{dR}{dt}$$
$$= 4\pi R^2 \frac{dR}{dt}$$

Now at $R = 100$,

$$\frac{dV}{dt} = 4\pi \times 100^2 \times (-10)$$
$$= -4\pi \times 10^5\text{mm}^3 \text{ per second.}$$

(iii) A spherical bubble is expanding so that its volume increases at the constant rate of 70 mm^3 per second. What is the rate of increase of its surface area when the radius is 10 mm?

Let V be the volume of the bubble.

Then $\frac{dV}{dt} = 70$.

Now $V = \frac{4}{3}\pi R^3$; $\therefore\ \frac{dV}{dt} = \frac{dV}{dR} \cdot \frac{dR}{dt} = 4\pi R^2 \frac{dR}{dt}$ (1)

Also $A = 4\pi R^2$; $\therefore\ \frac{dA}{dt} = 8\pi R. \frac{dR}{dt}$ (2)

Thus $\frac{dA}{dt} = \frac{8\pi R}{4\pi R^2} \cdot \frac{dV}{dt}$,

$$\frac{dA}{dt} = \frac{2}{R}\frac{dV}{dt} = \frac{2 \times 70}{10}$$

Now when $R = 10$, $\frac{dA}{dt} = 14\ \text{mm}^2$ per second.

This example illustrates the elimination of $\frac{dR}{dt}$ between (1) and (2) but the simple calculation of $\frac{dR}{dt}$ itself from (1) should not be overlooked.

14.2 Students initially should sketch the curve $y = Ae^{kx}$ for the various values of A and k, both positive and negative.

Let N be a population. Note that $N(t)$ is a function of time. Assume that the birth and death rates at any one time are proportional to N, so that the rate of change of population is given by

$$\frac{dN}{dt} = kN. \qquad (3)$$

k is assumed to be constant and is called the *growth rate*; it might be different for different species or locations. The rate is often given as a percentage, but in (3) it should be expressed as a decimal or fraction. If $k > 0$, the birth rate is larger than the death rate, while if $k < 0$ the birth rate is smaller than the death rate. If for a particular species the birth and death rates are equal, then $k = 0$, N is constant and the population is static.

Derivation of the solution of (3) is not required. Use direct substitution of $N = Ae^{kt}$ (A fixed) to demonstrate that it satisfies the equation for every choice of A, so that A fulfils the role of a constant of integration. The idea of an 'initial population' $N(0)$ should be introduced. It is then clear that $A = N(0)$, so that A is completely determined by the initial population. Since the subsequent population at a time t later is given by

$$N(t) = N(0)e^{kt},$$

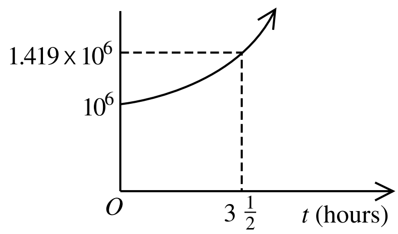
it is also completely determined.

Populations $N(t)$ should be graphed as functions of t for various values of A and k, as shown in the following examples.

(i) The growth rate per hour of a population of bacteria is 10% of the population. At $t = 0$ the population is 1.0×10^6. Sketch the curve of population after $3\frac{1}{2}$ hours, correct to 4 significant figures.

In this case $\frac{dN}{dt} = 0.1\,N$ so that $N = Ae^{0.1t}$, where t is in hours.

At $t = 0$, $N = 10^6$, so that $A = 10^6$.



After $3\frac{1}{2}$ hours the population is

$$N = 10^6 \times e^{0.35}$$
$$\approx 1.419 \times 10^6.$$

Note: The term 'growth rate per hour' is a widely used method of indicating that the time is to be measured in hours. The value of k is *not* a measure of an average rate of increase over a period of one hour ($N(1) \neq 1.1A$); rather, it indicates the instantaneous rate of increase of the population.

(ii) On an island, the population in 1960 was 1732, and in 1970 it was 1260. Find the annual growth rate to the nearest percent assuming it is proportional to the population. In how many years will the population be half that in 1960?

In this case $N = Ae^{kt}$. At $t = 0$, $N = 1732$, therefore $N = 1732e^{kt}$.
At $t = 10$, $N = 1260$, and hence

$$\begin{aligned} e^{10k} &= 1260/1732, \\ k &= \tfrac{1}{10} \log_e (1260/1732) \\ &\approx -0.0318 \\ &= -3\% \text{ (to the nearest percent.)} \end{aligned}$$

The population has halved at the time T years if

$e^{kT} = 0.5$. Thus $kT = \log_e 0.5$ and

$$\begin{aligned} T &= \frac{\log_e 0.5}{\frac{1}{10}\log_e(1260/1732)} \\ &= 21.79. \end{aligned}$$

$\therefore$ the population has halved in about 22 years.

The concept of exponential growth has been applied above to populations but could equally well be applied to depletion of natural resources, industrial production, inflation etc.

E

(iii) For 3 Unit students, consider also the equation

$$\frac{dN}{dt} = k(N - P),$$

where k and P are constants. First note that one solution of this equation is $N = P$. Direct substitution shows that the solution of this equation may be written in the form

$$N = P + Ae^{kt},$$

where A is an arbitrary constant. Some numerical examples should be undertaken, determining A and/or k from given initial conditions. It should be noted that whenever $k < 0$, the population N tends to the limit P as $t \to \infty$, irrespective of the initial conditions. The case $k > 0$ should also be discussed.

14.3 Velocity is defined as the rate of change of displacement, and acceleration as the rate of change of velocity. The notations $\frac{dx}{dt}$, $\dot{x}$, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $\ddot{x}$ should be introduced and used. Examples should concentrate on simple applications including physical descriptions of the motion of a particle given its distance from an origin, its velocity or its acceleration as a function of time. The significance of negative displacements, velocities and accelerations should be clearly understood. Some examples illustrating these points are now given.

(i) The acceleration a ms^{-2} of a moving particle is given after t seconds by $a = -3t$. Initially the particle is located at $x = 0$ and its velocity $v = 2$ ms^{-1}. Find the velocity v and displacement x as functions of time. Determine when the particle is at rest and when it returns to the origin at 0. Sketch x as a function of time. Describe the motion.

If $a = -3t$, $v = \int a\,dt = -\frac{3}{2}t^2 + A$.

At $t = 0$, $v = 2$ and therefore $A = 2$.

$v = -\frac{3}{2}t^2 + 2$.

Now $x = \int v\,dt = \int(-\frac{3}{2}t^2 + 2)dt$.

$\therefore\ x = -\frac{1}{2}t^3 + 2t + C$.

But at $t = 0$, $x = 0$, $\therefore\ C = 0$.

$x = -\frac{1}{2}t^3 + 2t$.

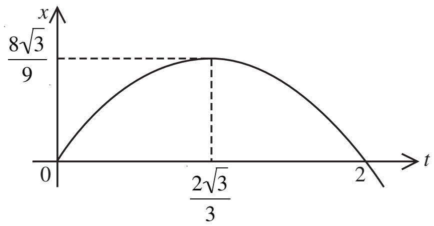
The particle is at rest when $v = 0$, ie when $t^2 = \frac{4}{3}$, so that $t = \frac{2\sqrt{3}}{3}$ seconds.

The particle returns to 0 when $x = 0$.

$t(2 - \frac{1}{2}t^2) = 0$

$\frac{1}{2}t(2 - t)\ (2 + t) = 0$.

Thus $x = 0$ at $t = 0, \pm 2$ and the particle returns to the origin after 2 seconds (since $t = -2$ occurs before the start of the motion).



Note that at $t = \frac{2\sqrt{3}}{3}$,

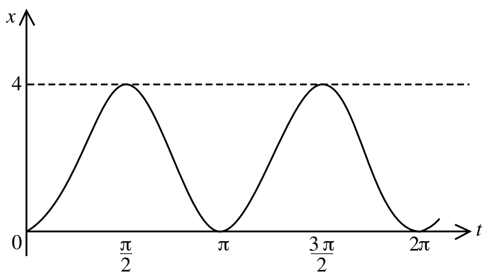
$x = \frac{8\sqrt{3}}{9}$.

The particle was initially located at the origin, was travelling to the right (since $v > 0$) and was slowing down (since $v > 0$, $a < 0$). It stopped after $\frac{2\sqrt{3}}{3}$ seconds, $\frac{8\sqrt{3}}{9}$ to the right of 0. It then travels back to the left (since $v < 0$ when $t > \frac{2\sqrt{3}}{3}$) with increasing speed (since $v < 0$, $a < 0$), passing through the origin after 2 seconds. It continues to travel left (since $v \neq 0$ for all other values of t).

(ii) A particle moves in a straight line. At time t seconds its distance x metres from a fixed point O in the line is given by

$x = 2 - 2\cos 2t$.

Sketch the graph of x as a function of t. Find the times when the particle is at rest and the position of the particle at those times. Describe the motion.



The velocity is given by $v = \frac{dx}{dt} = 4 \sin 2t$.

$\therefore$ $v = 0$ when $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$ and at these times, $x = 0, 4, 0, 4, \ldots$.

The particle is initially at 0 and is at rest. It starts to travel to the right with increasing speed for $\frac{\pi}{4}$ seconds. It then slows down and stops after $\frac{\pi}{2}$ seconds at a position 4 units to the right of 0. It travels back to the left arriving at 0 after an additional $\frac{\pi}{2}$ seconds. It continues to oscillate between $x = 0$ and $x = 4$ taking π seconds for one complete oscillation. (It should be noted that the above description may be obtained directly from the displacement-time graph).

Note: It is suggested that students be led gradually into a full description of the particle's motion by first attempting to describe the motion only from consideration of the displacements of the particle at various times.

E

14.3 If it is also possible to give v as a function of x, then, using the function of a function rule,

$$\frac{dv(x)}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}, \text{ but } \frac{dx}{dt} = v, \text{ so}$$

$$\frac{dv(x)}{dt} = \frac{dv}{dx} \cdot v = \frac{1}{2}\frac{dv^2}{dx}.$$

Problems involving the solutions of equations of the forms

$$\frac{d^2x}{dt^2} = f(x) \text{ and } \frac{d^2x}{dt^2} = \frac{1}{2}\frac{dv^2}{dx} = g(x)$$

should be considered. Eg, find x as a function of t given that

$$\frac{1}{2}\frac{dv^2}{dx} = -\frac{1}{2}e^{-x}, \; v = 1, x = 0 \text{ at } t = 0.$$

By integration, $v^2 = e^{-x} + C$.

But $v = 1$ when $x = 0$. $\therefore$ $C = 0$.

$v^2 = e^{-x}$,

$\therefore$ $v = \pm\, e^{-\frac{1}{2}x}$.

To decide which expression for v is relevant for this motion it will be necessary to examine the given data.

At $x = 0$, $v = 1 > 0$.

But $e^{-\frac{1}{2}x} > 0$

$\therefore$ $v = e^{-\frac{1}{2}x}$.

Hence $\frac{dx}{dt} = e^{-\frac{1}{2}x}$.

Using the rule for derivatives of inverse functions (see Topic 15.1)

$\frac{dx}{dt} = e^{-\frac{1}{2}x}$, hence $t = 2e^{\frac{1}{2}x} - 2$ and

$x = 2\log_e(1 + t/2)$.

E

The *Motion of a Projectile*. The equations of motion of a particle projected vertically upwards should be derived.

The two-dimensional motion of a projectile with the initial conditions that at $t = 0$, $x = y = 0$, $u = \frac{dx}{dt} = V\cos\alpha$, $v = \frac{dy}{dt} = V\sin\alpha$, results in the expressions that, at time t, $x = Vt\cos\alpha$, $y = Vt\sin\alpha - \frac{1}{2}gt^2$. This pair of equations gives a *parametric representation* of the 'flight parabola'. Here V is the initial speed and α the angle of projection. The cartesian equation of the flight parabola is

$$y = x\tan\alpha - \tfrac{1}{2}gx^2/(V^2\cos^2\alpha).$$

The range should be derived for a projectile fired on a horizontal plane. The maximum range on a horizontal plane is V^2/g when $\alpha = 45°$.

E

14.4 It follows immediately from the given equation that

$$v = \dot{x} = -an\sin(nt + \alpha)$$

and that

$$\dot{v} = \ddot{x} = -an^2\cos(nt + \alpha) = -n^2x.$$

Graphs of x, $\dot{x}$ and $\ddot{x}$ as functions of t should be sketched and the relationships between zero, minimum and maximum values of the three quantities noted. The physical significance of the parameters a, n and α should be understood, as should the terms amplitude, frequency, period and phase.

The differential equation of the motion may be interpreted as describing the motion of a particle acting under a force directed towards the origin O and proportional to the distance from O. This occurs in practice where a particle oscillates about an equilibrium position (as, for example, in the motion of a pendulum bob or of a mass attached to a spring, or the bobbing motion of a buoy).

Note that from the expression for x and v,

$$v^2 + n^2x^2 = a^2n^2,$$

a positive constant (ie independent of t). Notice also that if n is given, then a and α may be determined from a knowledge of x and v at any given time t_0:

a directly from the above equation and α from either of the two expressions for x and v.

Extension to the case

$$x = b + a\cos(nt + \alpha)$$

and the corresponding equation

$$\ddot{x} = -n^2(x - b)$$

describing simple harmonic motion about the position $x = b$.

E

An alternative treatment of simple harmonic motion begins with the differential equation $\ddot{x} = -n^2x$ and then uses the inverse trigonometric functions to derive a solution of the form $x = a \cos(nt + \alpha)$. A rigorous treatment along these lines is difficult because x is not an invertible function of t (and v is not a function of x) unless the range of x is restricted. It can be observed that

$$x = x_0 \cos(n(t - t_0)) + \frac{v_0}{n} \sin(n(t - t_0))$$

is a solution of the equation, which satisfies the additional conditions $x(t_0) = x_0$, $v(t_0) = v_0$. Questions of uniqueness of solution are outside the scope of the syllabus.

E

15. Inverse Functions and the Inverse Trigonometric Functions

15.1 Suppose that $y = f(x)$ is a continuous *increasing* function in the domain $a \le x \le b$ (ie $f(\beta) > f(\alpha)$ for all $\beta > \alpha$ in this domain). To each value y in $f(a) \le y \le f(b)$ there corresponds a *unique* value (namely, the x such that $y = f(x)$) and we may write $x = g(y)$, where the function g has domain $f(a) \le y \le f(b)$.

The two relations $y = f(x)$ and $x = g(y)$ are equivalent and are represented by the same graph in the x, y plane. These two relations are *mutually inverse functions* in the sense that

for $a \le x \le b$, $g(f(x)x) = g(y) = x$,

for $f(a) \le y \le f(b)$, $y = f(x) = f(g(y))$;

accordingly the notation f^{-1} is commonly used for the function inverse to f.

If we express the function f^{-1} in the conventional form $y = f^{-1}(x)$, its graph is obtained from that of $y = f(x)$ by reflection in the line $y = x$. The domain of $y = f^{-1}(x)$ is the range of $y = f(x)$ and vice versa.

Care must be taken to distinguish $f^{-1}(x)$ from $(f(x))^{-1} = \frac{1}{f(x)}$.

Simple examples of mutually inverse functions, which could be used to introduce this topic, are:

(i) $y = x^3$, all real x; $y = x^{\frac{1}{3}}$, all real x.

(ii) $y = e^x$, all real x; $y = \log_e x$, $x > 0$.

The problem of defining an inverse function when the equation $y = f(x)$ has more than one solution x for a given y should be discussed; the case $y = x^2$ is a useful illustration.

Differentiation. If in addition f is a differentiable function of x then (since a tangent line to $y = f(x)$ is also a tangent line to $x = g(y)$) g is a differentiable function of y, and (since the relevant angles of inclination are complementary)

$$\frac{dx}{dy} = 1/\left[\frac{dy}{dx}\right] \text{ or } \frac{dy}{dx} \cdot \frac{dx}{dy} = 1,$$

which may be formally obtained from the definition of the derivative.

Example If $y = x^3$, $\frac{dy}{dx} = 3x^2 = 3y^{\frac{2}{3}}$. Hence $\frac{dx}{dy} = \frac{1}{3y^{\frac{2}{3}}} = \frac{d}{dy}(y^{\frac{1}{3}})$.

E

15.2 The inverse function $y = \sin^{-1}x$ should be defined with domain $-1 \le x \le 1$ and range $-\pi/2 \le y \le \pi/2$. The general solution of the equation $\sin \theta = b$, for $|b| \ge 1$, is then expressible as $\theta = n\pi + (-1)^n \sin^{-1}b$. The functions $y = \cos^{-1}x$ $(0 \le y \le \pi)$ and $y = \tan^{-1}x$ $(-\frac{1}{2}\pi < y < \frac{1}{2}\pi)$ should also be defined, and the general solution of the equations $\cos \theta = b$ $(|b| \le 1)$ and $\tan \theta = b$ should be obtained.

E

15.4 For example, properties such as:

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $\tan^{-1}(-x) = -\tan^{-1}x$;

(ii) $\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi$.

E

15.5 The derivatives of $\sin^{-1}(x/a)$, $\cos^{-1}(x/a)$ and $\tan^{-1}(x/a)$ should be obtained. The corresponding integrations should be known.

E

16. Polynomials

16.1 A real polynomial $P(x)$ of degree n is an expression of the form

$$P(x) = p_0 + p_1x + \ldots + p_{n-1}x^{n-1} + p_nx^n \ (p_n \ne 0),$$

where the real numbers $p_0, \ldots, p_n$ are called the coefficients of $P(x)$ and for convenience will usually be chosen to be integers. The degree of $P(x)$ is that of the highest power of x occurring with non-zero coefficient.

$P(x)$ is defined for all real x and is a (continuous and) differentiable function of x. The equation $P(x) = 0$ is called a polynomial equation of degree n, and those real numbers x which satisfy the equation are called real roots of the equation or real zeros of the corresponding polynomial. Examples should be given illustrating cases where one or more real roots occur and where none occur.

Graphs of simple polynomials should be drawn, using all the techniques available. The following useful facts should be noted.

(i) For very large $|x|$, $P(x) \approx p_nx^n$

(ii) A polynomial of odd degree always has at least one real zero.

(iii) At least one maximum or minimum value of P occurs between any two distinct real zeros.

16.2 Long division of one polynomial by another should be discussed and illustrated by examples using linear or quadratic divisors. The division process should be expressed as an identity:

$$P(x) = A(x)Q(x) + R(x),$$

where $A(x)$ is the divisor, $Q(x)$ the quotient and $R(x)$ the remainder. The degree of $R(x)$ must be less than that of $A(x)$. With this condition satisfied, it may be stated that $Q(x)$ and $R(x)$ are then unique.

E

Rational functions should be defined as ratios of polynomials and the division process also expressed in the form

$$\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}.$$

The remainder theorem, which states that the remainder when $P(x)$ is divided by $x - a$ is $P(a)$, and the factor theorem, which states that if $P(a) = 0$ then $x - a$ is a factor of $P(x)$, both follow from the identity and the condition on $R(x)$.

The following results should be obtained.

(i) If $P(x)$ has k distinct real zeros $a_1, \ldots, a_k$, then $(x - a_1) \ldots (x - a_k)$ is a factor of $P(x)$.

(ii) If $P(x)$ has degree n and n distinct real zeros $a_1, \ldots, a_n$, then $P(x) = p_n(x - a_1) \ldots (x - a_n)$.

(iii) A polynomial of degree n cannot have more than n distinct real zeros.

(iv) A polynomial of degree at most n, which has more than n distinct real zeros, is the zero polynomial (ie the polynomial in which $p_0 = p_1 = \ldots = p_n = 0$).

16.3 The convention of double and multiple roots should be explained and the results above extended to show that a polynomial equation of degree n has at most n real roots (and may have none).

The relation between the coefficients and the roots (if they exist) of the quadratic equation $ax^2 + bx + c = 0$ should now be derived using the identity

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

The corresponding relations for cubic equations should also be derived and the general result indicated. Particular examples should not involve polynomial equations of degree above four.

16.4 Discussion may be confined to the following two methods.

(i) Halving the interval. Suppose we have two values of x, say $x = x_1$ and $x = x_2$, such that the polynomial $P(x)$ is positive for $x = x_1$, ie $P(x_1) > 0$, and is negative for $x = x_2$, ie $P(x_2) < 0$. Since $P(x)$ is a continuous function, there is a root of $P(x)$ in the interval $x_1 < x < x_2$.

Now compute the midpoint $x_3 = \frac{1}{2}(x_1 + x_2)$ and the corresponding polynomial value $P(x_3)$. If $P(x_3) = 0$, x_3 is the desired root. If $P(x_3) > 0$, we replace x_1 by x_3 and repeat the process using x_3 and x_2. If $P(x_3) < 0$, we replace x_2 by x_3 and repeat the process.

E (ii) Newton's method. Suppose z is close to a root of $P(x) = 0$. The tangent to $y = P(x)$ at $x = z$ has the equation $y - P(z) = P'(z)\,(x - z)$. This tangent intersects the x-axis at $x = z - P(z)/P'(z)$. If the original value of z was sufficiently close to the desired root, and if certain other conditions are satisfied, the new value x is even closer. We repeat the process to converge in general to the desired root. Newton's method is in principle faster (requires fewer steps for a given accuracy) than halving the interval, but some care must be exercised in applying it. A check that the values obtained do appear to be approaching a root should be made by calculating the corresponding function values.

Both of these methods may be applied to find approximate roots of equations involving other types of functions.

E

17. Binomial Theorem

17.1 The binomial expansion is introduced by example, using $n = 1,2,3,4$. The Pascal triangle is constructed.

We observe that for any integral n, no matter how large, $(1 + x)^n$ is a polynomial of degree n in the variable x. The (so far unknown) coefficients in this polynomial must be labelled by some symbol; we choose 'C' for 'coefficient' and write the power of x as a right subscript k, the power n as a left superscript, ie ${}^nC_k$ is by definition the coefficient of x^k in $(1 + x)^n$:

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x + \dots\ {}^nC_nx^n.$$

We extend this identity to an expansion of $(a + u)^n$ by putting $x = u/a$, and multiplying both sides by a^n; the result is:

$$(a + u)^n = {}^nC_0a^n + {}^nC_1a^{n-1}u + {}^nC_2a^{n-2}u^2 + \dots + {}^nC_nu^n.$$

Proof of the Pascal triangle relations

There are two separate relations, (i) for the outer coefficients, and (ii) for the others.

In the above expansion for $(a + u)^n$, put $a = 1$, $u = 0$ on both sides to get ${}^nC_0 = 1$. Put $a = 0$, $u = 1$ on both sides to get ${}^nC_n = 1$.

Next we write down $(1 + x)^{n-1}$ and below it, the same expression multiplied by x; adding them together, and collecting terms of the same degree in x, we obtain

$$(1 + x)^{n-1} + x\,(1 + x)^{n-1} = {}^{n-1}C_0 + ({}^{n-1}C_0 + {}^{n-1}C_1)x +$$

$$({}^{n-1}C_1 + {}^{n-1}C_2)x^2 + \dots + ({}^{n-1}C_{n-2} + {}^{n-1}C_{n-1})x^{n-1} + {}^{n-1}C_{n-1}x^n.$$

However, the left side is obviously equal to $(1 + x)^n$. We now use the fact (which we have proved as a theorem only for the special case of second degree polynomials) that two polynomials are equal for all values of x if and only if all corresponding coefficients are equal. Comparing the right side above with the earlier expansion of $(1 + x)^n$, we immediately deduce:

$${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k \text{ for } 1 \le k \le n - 1.$$

E 17.2 **Proof by Mathematical Induction of the formula for ^nC_k**

Let us consider $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$. The ratios of successive coefficients are: $4/1 = n/1$; $6/4 = 3/2 = (n - 1)/2$; $4/6 = 2/3 = (n - 2)/3$; and finally, $1/4 = (n - 3)/4$. Thus the coefficients in this formula are, going from left to right,

$$1,\ \frac{n}{1},\ \frac{n}{1}\cdot\frac{n-1}{2},\ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3},\ \text{and}\ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}$$

This leads us to guess at the general formula:

statement $S(n)$: $^nC_k = \dfrac{n(n-1)\ (n-2)\ldots(n-k+1)}{1\times2\times3\times\ldots\times k}$ for $1 \le k \le n$.

The remaining coefficient, $^nC_0$, has already been shown to equal 1 in 17.1. It should also be noted at this stage that our guess gives the correct, and already proved, answer for ^nC_n. We now use mathematical induction to prove this result.

Step 1. Statement S (2) is true, by inspection of $(1 + x)^2$.

Step 2. We assume statement $S(n - 1)$ to be true, and deduce from it the truth of $S(n)$.

The easiest case is $k = 1$. We assume the truth of $^{n-1}C_1 = (n - 1)/1$, which is part of statement $S(n - 1)$ and we already know, from 17.1, that $^{n-1}C_0 = 1$. Adding these two, and using the Pascal triangle relation proved in 17.1, we obtain $^nC_1 = 1 + (n - 1) = n = n/1$, in agreement with statement $S(n)$.

For $k \ge 2$, we have $k - 1 \ge 1$ and we assume the truth of the relations:

$$^{n-1}C_{k-1} = \frac{(n-1)\ (n-2)\ (n-3)\ldots(n-k)}{1\times2\times3\times\ldots\times k} \quad \text{and}$$

$$^{n-1}C_{k-1} = \frac{(n-1)\ (n-2)\ldots(n-k+1)}{1\times2\times3\times\ldots\times(-1)},$$

both of which are part of statement $S(n-1)$. We add these two numbers and use the Pascal triangle relation to get

$$^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k = \frac{(n-1)\ (n-2)\ldots(n-k+1)}{1\times2\times3\times\ldots\times(-1)}\left[1 + \frac{n-k}{k}\right].$$

Since the square bracket equals n/k, this is precisely the same as statement $S(n)$, which is hereby proved for all $k \ge 2$, and has been proved for $k = 1$ above. The proof by induction is therefore complete.

17.3 The student should be able to use the general formula

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

For example, find the coefficient of x^3 when $(x^2 - 2/x)^{3N}$ is expanded in powers of x, given that N is a positive integer.

The typical term is $\binom{3N}{k} x^{2k} (-2/x)^{3N-k}$, in which the power of x is $3k - 3N$. Taking $k = N + 1$, we find that the required coefficient is

$$(-2)^{2N-1}\binom{3N}{n+1}.$$

E

When a and b are positive, consideration of the ratio of successive terms in the expansion of $(a + b)^n$ makes it easy to determine the greatest term in that expansion, eg

find the greatest coefficient in the expansion of $(3 + 5x)^{20}$.

Writing $(3 + 5x)^{20} = \sum_{k=0}^{20} t_k x^k$, we have $\frac{t_{k+1}}{t_k} = \frac{5}{3} \cdot \frac{(20-k)}{k+1}$, which exceeds one for $1 \le k \le 12$ and is less than one thereafter. Hence the greatest coefficient is $t_{13} = \binom{20}{13} 5^{13} 3^7$.

The substitution, $x = 1$, in the expansion of $(1 + x)^n$ gives the formula $2n = \sum_{k=0}^{n} \binom{n}{k}$. (This can, of course, be given a direct combinatorial interpretation as the equality of two methods of enumerating the subsets of a set of n elements.) We may use functional properties of $(1 + x)^n$ to sum other finite series involving binomial coefficients. For example, consideration of the coefficient of x^n on each side of the identity, $(1 + x)^n(1 + x)^n \equiv (1 + x)^{2n}$, gives the formula

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

Differentiation or integration is also possible, eg differentiate both sides of the identity

$$(1 + x)^{2n} \equiv \sum_{k=0}^{2n} \binom{2n}{k} x^k,$$

and show that

$$\sum_{k=1}^{2n} k \binom{2n}{k} = n4^n.$$

E

18. Permutations, Combinations and Further Probability

18. It is natural to introduce this topic in the context of enumerating the outcomes of random experiments, so it is preferable that students have previously developed some familiarity with the material on probability theory.

18.1 We start with n different objects and choose k of them one after another. (An ordered sample without replacement.) Since the first can be chosen in n ways, then the next in $(n - 1)$ ways and so on, this can be done in $n.(n - 1).(n - 2) \ldots (n - k + 1)$ ways. In other words, we have found the number of ways of arranging n objects k at a time and this number is also written ${}^nP_k$. In the special case where $k = n$, we are dealing with the total number of arrangements or permutations of n distinct objects. This number is
$n! = {}^nP_n = n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1.$

E

Using factorial notation and the convention $0! = 1$,

$${}^{n}P_{k} = n!/(n-k)! \text{ for } k = 0, 1, \ldots, n.$$

Now suppose we ignore the order in which the k objects were chosen from our pool of n objects. We wish to find the number, ${}^{n}C_{k}$, of ways of choosing a collection of k objects from a collection of n objects. One way to do this is to note that choosing an ordered sample is equivalent to choosing an unordered sample then ordering it. An ordered sample can be chosen in ${}^{n}P_{k}$ ways, the second procedure can be carried out in ${}^{n}C_{k} \times k!$ ways. Therefore

$${}^{n}C_{k} = {}^{n}P_{k}/k! = \frac{n!}{(n-k)!\ k!}.$$

For instance, in Lotto, the number of ways of selecting six of the numbers 1 to 40 is ${}^{40}C_{6}$. In TAB betting the trifecta pays on the first three horses in correct order, the quinella pays on the first two horses in either order. In a 12 horse race the number of possible trifecta combinations is ${}^{12}P_{3}$ and the number of possible quinella combinations is ${}^{12}C_{2}$. Note that in the case of Lotto the enumerated outcomes are equally likely, while in the other cases the outcomes are (presumably) not equally likely.

At this stage it is advisable to make the connection with the work on the binomial expansion. Since the coefficient of x^k in the expansion of $(1 + x)^n$ is equal to the number of ways of choosing k xs from the n factors $(1 + x)$, this coefficient is (by the above) just

$${}^{n}C_{k} = \frac{n(n-1)\ldots(n-k+1)}{1 \times 2 \times \ldots \times k}.$$

giving an alternative proof of the result in 17.2.

It is also interesting to give combinatorial proofs of the relations

$${}^{n}C_{k} = {}^{n}C_{n-k}, \quad {}^{n+1}C_{k} = {}^{n}C_{k} + {}^{n}C_{k-1}.$$

When we select k objects from n, we discard $n - k$ objects. Hence the number of ways of selecting k objects from n equals the number of ways of selecting $n - k$ objects from n, and this shows that

$${}^{n}C_{k} = {}^{n}C_{n-k}.$$

Let us add a new object to our original pool of n objects. Now we select k objects from the new pool (as we can in ${}^{n+1}C_{k}$ ways). Either our selection contains the new object or it does not. In the first case we are effectively choosing $k - 1$ objects from the original n (and this arises in ${}^{n}C_{k-1}$ ways) while in the second case we are choosing k objects from the original n (and this occurs in ${}^{n}C_{k}$ ways). It follows that

$${}^{n+1}C_{k} = {}^{n}C_{k} + {}^{n}C_{k-1}.$$

The following are typical problems.

In choosing three letters from the word PROBING, and assuming each choice is equally likely, what is the probability of choosing just one vowel?

The total number of choices is ${}^{7}C_{3} = 35$. The number of ways of choosing one vowel and two consonants is ${}^{2}C_{1} \times {}^{5}C_{2} = 20$ .Thus the required probability is 4/7.

E How many numbers greater than 5000 can be formed with the digits 2, 3, 5, 7, 9 if no digit is repeated?

First there are ${}^5P_4$ five-digit numbers. We can also choose a four-digit number, but then the first digit must be 5, 7 or 9, so there are $3 \times {}^4P_3$ possible numbers of this type. The total number of possibilities is 120 + 72 = 192.

In how many ways can the letters of EERIE be arranged in line?

5 distinct letters admit 5! arrangements and 3 distinct letters admit 3! arrangements. Thus the answer is 5!/3!.

In how many ways can the numbers 1, 2, 3, 4, 5, 6 be arranged around a circle? How many of these arrangements have at least two even numbers together?

The first number may be placed arbitrarily. The next can be placed in 5 ways, the next in 4 ways, the next in 3 ways and so on; there are 5.4.3.2 = 120 arrangements. There are always at least two adjacent even numbers unless the numbers are arranged odd, even, odd, even, odd, even as we go round. Place the odd numbers in 2 ways. Now insert the even numbers in the available gaps in 3! ways. This gives 12 alternating arrangements and hence 108 = 120 – 12 arrangements of the required kind.

18.2 Suppose that a coin gives heads (H) with probability p and tails (T) with probability $q = 1 - p$. Representing the outcomes of three tosses on a tree diagram spreading vertically downwards, we find that the bottom row has the entries HHH, HHT, HTH, HTT, THH, THT, TTH, TTT with respective probabilities p^3, p^2q, p^2q, pq^2, p^2q, pq^2, pq^2, q^3. The number X of heads appearing in these three tosses has a frequency distribution given by $P(X = 0) = q^3$, $P(X = 1) = 3pq^2$, $P(X = 2) = 3p^2q$, $P(X = 3) = p^3$ and it is easy (and helpful) to draw a histogram for various values of p.

It is straightforward to generalise to the case of n tosses, where the outcomes are represented by strings of length n using the letters H, T and each string arises with probability p^rq^{n-r}, where r is the number of times H appears. Because there are $\binom{n}{r}$ strings with r occurrences of H and $n - r$ occurrences of T, we see that

$$P(X = r) = \binom{n}{r} p^rq^{n-r},$$

where X denotes the number of heads which appear in n tosses. X is said to have a *binomial distribution*. (Observe that the statement

$$\sum_{r=0}^{n} P(X = r) = 1$$

can be rewritten in the form

$$\sum_{r=0}^{n} \binom{n}{r} p^rq^{n-r} = (p + q)^n = 1.)$$

E The binomial distribution can be used to model repeated trials of any experiment with precisely two outcomes. The probability p may be known in advance, eg sampling with replacement from a box with two red and three white balls ($p = 2/5$ if H represents drawing a red ball); or estimated from frequency considerations, eg guessing the sex of a baby (p is slightly greater than $\frac{1}{2}$ if H represents the birth of a girl).

The following are typical examples.

It is known that $x\%$ of the bolts produced by a machine are faulty. What is the probability that in a random sample of 4 bolts:

(a) no bolts are defective?

(b) precisely one bolt is defective?

(c) at most, two bolts are defective?

(Express all answers in the form of $10^{-8}R(x)$, where R is a polynomial which need not be simplified.)

Let $p = x/100$ denote the probability that a bolt is defective and $q = 1 - p$.

Then the required probabilities are respectively $\binom{4}{0} q^4$, $\binom{4}{1} pq^3$, and the sum of the first two together with $\binom{4}{2} p^2q^2$. Thus we can write the answers as

(a) $10^{-8} (100 - x)^4$,

(b) $10^{-8} \times 4x (100 - x)^3$

(c) $10^{-8} \times \{(100 - x)^4 + 4x(100 - x)^3 + 6x^2 (100 - x)^2\}$.

On the average, batsmen in a certain cricket team make a scoring shot on every third ball. Estimate how many six-ball overs with precisely two scoring shots occur in a thousand overs of batting by that team.

We take $p = \frac{1}{3}$ to represent the probability of a scoring shot on a given ball.

The probability that a random over contains precisely 2 scoring shots is $\binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$. We multiply this by 10^3 and (rounding off to the nearest integer) estimate that there are 329 overs.